VISION

ON THE APPLICATION OF VECTOR ALGEBRA TO PROJECTIVE GEOMETRY

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1. Introduction. The study of projective geometry by vector methods was originated in principle by Grassman in his *Ausdehnungslehre*. It was carried out in detail by E. Study in his *Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vektorenrechnung* (1923), and appears also in the book of H. G. Forder, *The Calculus of Extension* (1941). Since this work does not appear to be generally known, it is the purpose of this paper to publicize it by giving a connected treatment of some of the chief theorems of two-dimensional projective geometry based upon this vector approach.

We define $\mathbf{a}=(a_1, a_2, a_3)$ to be a vector representing a point in the projective plane. The usual definitions and identities of vector algebra hold, and we give the most important ones below:

$$\mathbf{a}+\mathbf{b}=(a_1+b_1, a_2+b_2, a_3+b_3) \tag{1}$$

$$\lambda\mathbf{a}=(\lambda a_1, \lambda a_2, \lambda a_3). \tag{2}$$

For $\lambda\neq 0$, this represents the same point as the vector $\mathbf{a}$.

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3 \tag{3}$$

$$\mathbf{a}\times\mathbf{b}=(a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1). \tag{4}$$

We have then, besides the well known associative, commutative, and distributive laws:

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=[\mathbf{abc}] \tag{5}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a} \tag{6}$$

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}'\times\mathbf{b}')&=[\mathbf{aa'b'}]\mathbf{b}-[\mathbf{ba'b'}]\mathbf{a}\\&=[\mathbf{abb'}]\mathbf{a}'-[\mathbf{aba'}]\mathbf{b}'\end{aligned} \tag{7}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}'\times\mathbf{b}')=(\mathbf{a}\cdot\mathbf{a}')(\mathbf{b}\cdot\mathbf{b}')-(\mathbf{a}\cdot\mathbf{b}')(\mathbf{a}'\cdot\mathbf{b}). \tag{8}$$

This relation is known as Lagrange's identity. From (8), we derive:

$$\begin{aligned}[\mathbf{a}\times\mathbf{a}', \mathbf{b}\times\mathbf{b}', \mathbf{c}\times\mathbf{c}']&=[\mathbf{abb'}][\mathbf{a'cc'}]-[\mathbf{acc'}][\mathbf{a'bb'}]\\&=[\mathbf{bcc'}][\mathbf{b'aa'}]-[\mathbf{baa'}][\mathbf{b'cc'}]\\&=[\mathbf{caa'}][\mathbf{c'bb'}]-[\mathbf{cbb'}][\mathbf{c'aa'}]\end{aligned} \tag{9}$$

which will be used frequently later on.

We shall also introduce the vector $\boldsymbol{\alpha}=(\alpha_1, \alpha_2, \alpha_3)$ to represent the line coordinates of a line in the plane. As above $\lambda\boldsymbol{\alpha}=(\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3)$ represents the same line. We shall henceforth use Latin letters for points and Greek letters for lines.

2. Determination of points and lines in the plane. We see at once that:

(i) $\mathbf{a}\cdot\alpha=0$ is the condition for $\mathbf{a}, \alpha$ to be in incidence.

(ii) $\mathbf{a}\times\mathbf{b}=0$ is the condition for a and b to coincide. By 0 we mean the vector (0, 0, 0).

(iii) $\mathbf{a}\times\mathbf{b}\neq 0$ is the line joining a and b.

(iv) $[\mathbf{abc}]=0$ is the condition for $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ to be collinear. We also have the duals of (ii), (iii), and (iv). From these it follows that:

(v) $(\mathbf{a}\times\mathbf{b})\times\alpha=(\mathbf{a}\cdot\alpha)\mathbf{b}-(\mathbf{b}\cdot\alpha)\mathbf{a}$ is the point of intersection of the line joining $\mathbf{a}, \mathbf{b}$ with the line α;

(vi) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}'\times\mathbf{b}')=[\mathbf{aa}'\mathbf{b}']\mathbf{b}-[\mathbf{ba}'\mathbf{b}']\mathbf{a}$ is the point of intersection of the lines joining $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}', \mathbf{b}'$;

(vii) $(\mathbf{a}\times\mathbf{b})\cdot(\alpha\times\beta)=0$ or $(\alpha\cdot\mathbf{a})(\beta\cdot\mathbf{b})=(\alpha\cdot\mathbf{b})(\beta\cdot\mathbf{a})=0$ is the condition for the line joining $\mathbf{a}, \mathbf{b}$ to pass through the point of intersection of α, β;

(viii) $[\mathbf{a}\times\mathbf{a}', \mathbf{b}\times\mathbf{b}', \mathbf{c}\times\mathbf{c}']=0$ is the condition for the three lines joining $\mathbf{a}, \mathbf{a}'$; $\mathbf{b}, \mathbf{b}'$; $\mathbf{c}, \mathbf{c}'$ in pairs to be concurrent. From (9), we see that this condition may be written in three different ways.

In this manner, our list may be extended indefinitely. All may be dualized.

The advantage of the present representation in numerical problems is obvious; it saves considerable calculation. It can also be used to prove in a simple and concise manner many theorems of fundamental importance in plane projective geometry. We shall give a few examples of such theorems.

3. Diagonal triangle of a complete quadrangle. As a first illustration, let us prove the

THEOREM. *The three diagonal points of a complete quadrangle form a triangle.*

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be the vertices of a complete quadrangle so that no three of them are collinear. The diagonal points of the quadrangle are the intersections of opposite sides; namely,

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}), \qquad (\mathbf{a}\times\mathbf{c})\times(\mathbf{d}\times\mathbf{b}), \qquad (\mathbf{a}\times\mathbf{d})\times(\mathbf{b}\times\mathbf{c}).$$

These three points are not collinear; for, applying (9) twice, we find that their determinant is equal to

$$-2[\mathbf{bcd}][\mathbf{acd}][\mathbf{abd}][\mathbf{abc}]$$

which is not zero by (iv), since no three of the points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are collinear.

4. Desargues' theorem on perspective triangles. Next, let us take up

DESARGUES' THEOREM. *The corresponding vertices of two triangles lie on concurrent lines if and only if the points of intersection of corresponding sides meet in collinear points.*

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the vertices of one triangle and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$, the corresponding

vertices of the other. The condition for the lines joining corresponding vertices to be concurrent is

$$[\mathbf{a} \times \mathbf{a}', \mathbf{b} \times \mathbf{b}', \mathbf{c} \times \mathbf{c}'] = 0 \tag{10}$$

and the condition for the points of intersection of corresponding sides to be collinear is

$$[(\mathbf{b} \times \mathbf{c}) \times (\mathbf{b}' \times \mathbf{c}'), (\mathbf{c} \times \mathbf{a}) \times (\mathbf{c}' \times \mathbf{a}'), (\mathbf{a} \times \mathbf{b}) \times (\mathbf{a}' \times \mathbf{b}')] = 0. \tag{11}$$

Applying (9) again, we find that (10) is equivalent to

$$[\mathbf{abb}'][\mathbf{a}'\mathbf{cc}'] - [\mathbf{acc}'][\mathbf{a}'\mathbf{bb}'] = 0 \tag{12}$$

and (11) to

$$[\mathbf{abc}][\mathbf{a}'\mathbf{b}'\mathbf{c}'](\,[\mathbf{abb}'][\mathbf{a}'\mathbf{cc}'] - [\mathbf{acc}'][\mathbf{a}'\mathbf{bb}']) = 0. \tag{13}$$

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as well as $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are non-collinear, $[\mathbf{abc}][\mathbf{a}'\mathbf{b}'\mathbf{c}'] \neq 0$ and the conditions (12) and (13) are equivalent.

In particular, if $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ lie on the sides of the other triangle opposite to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively, we may put

$$\begin{aligned} \mathbf{a}' &= \lambda_1\mathbf{b} + \mu_1\mathbf{c}, \\ \mathbf{b}' &= \lambda_2\mathbf{c} + \mu_2\mathbf{a}, \\ \mathbf{c}' &= \lambda_3\mathbf{a} + \mu_3\mathbf{b}. \end{aligned} \tag{14}$$

Substituting into (12), we have the condition

$$(\lambda_1\lambda_2\lambda_3 - \mu_1\mu_2\mu_3)[\mathbf{abc}]^2 = 0 \tag{15}$$

or

$$\lambda_1\lambda_2\lambda_3 - \mu_1\mu_2\mu_3 = 0.$$

This gives the Theorem of Ceva. Its dual, the Theorem of Menelaus, is proved in a dual fashion.

5. Pole and polar w.r.t. a triangle. Again, let us prove the

THEOREM. *If three points, each lying on a side of a triangle, are collinear, the lines joining their harmonic conjugates w.r.t. the vertices of the triangle to the opposite vertices are concurrent and conversely.*

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the vertices of the triangle and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ be points on the sides opposite to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. If we assume that $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are given by (14), the condition for these points to be collinear is

$$[\lambda_1\mathbf{b} + \mu_1\mathbf{c}, \lambda_2\mathbf{c} + \mu_2\mathbf{a}, \lambda_3\mathbf{a} + \mu_3\mathbf{b}] = 0,$$

or

$$(\lambda_1\lambda_2\lambda_3 + \mu_1\mu_2\mu_3)[\mathbf{abc}] = 0. \tag{16}$$

If $\mathbf{a}''$, $\mathbf{b}''$, $\mathbf{c}''$ are the harmonic conjugates of $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ w.r.t. $\mathbf{b}, \mathbf{c}$; $\mathbf{c}, \mathbf{a}$; $\mathbf{a}, \mathbf{b}$ respectively, we may put

$$\mathbf{a}'' = \lambda_1\mathbf{b} - \mu_1\mathbf{c},$$
$$\mathbf{b}'' = \lambda_2\mathbf{c} - \mu_2\mathbf{a},$$
$$\mathbf{c}'' = \lambda_3\mathbf{a} - \mu_3\mathbf{b},$$

so that the condition for the lines joining $\mathbf{a}''$, $\mathbf{b}''$, $\mathbf{c}''$ to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to be concurrent is, by (15),

$$[\mathbf{a} \times \mathbf{a}'', \mathbf{b} \times \mathbf{b}'', \mathbf{c} \times \mathbf{c}''] = (\lambda_1\lambda_2\lambda_3 + \mu_1\mu_2\mu_3)[\mathbf{abc}]^2 = 0. \tag{17}$$

Since $[\mathbf{abc}] \neq 0$, the conditions (16) and (17) are equivalent.

6. Pappus' Theorem. We now prove

PAPPUS' THEOREM. *If a, b, c are three points on a line and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ three points on another, the three points of intersection of the pairs of lines joining $\mathbf{b}$ and $\mathbf{c}'$, $\mathbf{b}'$ and $\mathbf{c}$; $\mathbf{c}$ and $\mathbf{a}'$, $\mathbf{c}'$ and $\mathbf{a}$; $\mathbf{a}$ and $\mathbf{b}'$, $\mathbf{a}'$ and $\mathbf{b}$ are collinear.*

We have to show that

$$[(\mathbf{b} \times \mathbf{c}') \times (\mathbf{b}' \times \mathbf{c}), (\mathbf{c} \times \mathbf{a}') \times (\mathbf{c}' \times \mathbf{a}), (\mathbf{a} \times \mathbf{b}') \times (\mathbf{a}' \times \mathbf{b})] = 0 \tag{18}$$

which, according to (9), is equivalent to

$$[\mathbf{bb'c}][\mathbf{cc'a}][\mathbf{aa'b}][\mathbf{a'b'c'}] - [\mathbf{c'b'c}][\mathbf{a'c'a}][\mathbf{b'a'b}][\mathbf{abc}] = 0. \tag{18'}$$

Now since $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are collinear, $[\mathbf{abc}] = 0$ and similarly $[\mathbf{a'b'c'}] = 0$, and so (18′) is satisfied.

This result extends immediately to give a proof of

PASCAL'S THEOREM: *If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a}', \mathbf{b}', \mathbf{c}'$ are six points on a conic, the three points of intersection of the pairs of lines joining $\mathbf{b}$ and $\mathbf{c}'$, $\mathbf{b}'$ and $\mathbf{c}$; $\mathbf{c}$ and $\mathbf{a}'$, $\mathbf{c}'$ and $\mathbf{a}$; $\mathbf{a}$ and $\mathbf{b}'$, $\mathbf{a}'$ and $\mathbf{b}$ are collinear.*

Let us think of (18′) as an equation in the variable point $\mathbf{c}$. It appears at once that this equation is of the second degree in the coordinates of $\mathbf{c}$ and hence represents a conic. Further it is easy to see that the equation is identically satisfied when $\mathbf{c}$ is equal to any one of $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}'$, or $\mathbf{c}'$; and hence this is an equation of the conic passing through these last five points. Since the point $\mathbf{c}$ given in the theorem lies on this conic by hypothesis, (18′) is satisfied and the conclusion follows. This demonstration illustrates the fact that Pappus' Theorem is a degenerate form of Pascal's Theorem.

From (18′) we also have the following joint converse of these two theorems:

CONVERSE THEOREM: *If six points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a}', \mathbf{b}', \mathbf{c}'$ are so located that the three points of intersection of the pairs of lines joining $\mathbf{b}$ and $\mathbf{c}'$, $\mathbf{b}'$ and $\mathbf{c}$; $\mathbf{c}$ and $\mathbf{a}'$, $\mathbf{c}'$ and $\mathbf{a}$; $\mathbf{a}$ and $\mathbf{b}'$, $\mathbf{a}'$ and $\mathbf{b}$ are collinear, then*

(1) *if no three points are collinear, the six points lie on a non-degenerate conic*;
(2) *if three points* (*say* **a**, **b**, *and* **c**) *are collinear either: one or more of* **a′**, **b′**, **c′**, *lies on the line* **abc** *or*: **a′**, **b′**, *and* **c′** *are collinear.*

In interpreting these results it should be noted that the exact form of (18′) is not unique. Indeed it has 60 equivalent forms obtained from the one given by permuting any five of the given points. The equivalence of these forms can be shown directly by careful algebraic manipulation in the reduction of (18) to (18′) or from the fact that the equation of a conic through five points clearly is independent of the order in which these points are chosen.

MUSIC AND TERNARY CONTINUED FRACTIONS*

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1. Introduction. The most important interval in music is the octave, expressed by the ratio, 2:1. The number of notes within the octave has varied at different times and in different places. The pentatonic or black-key scale is very old and is widely distributed, occurring in such countries as Scotland and China. The heptatonic, diatonic, or white-key scale, in its various major and minor forms, is the foundation of our classic musical system. The duodecuple or chromatic scale includes both the white keys and the black keys of the piano, and has become increasingly important during the past two or three centuries.

After harmony was introduced into music during the late Middle Ages, major and minor triads emerged as the principal chords. The major triad, as *C E G*, was regarded with especial favor, because it occurs naturally in the harmonic series, as on bugles, and can be expressed by the simple ratios, 4:5:6. A system of tuning for the diatonic scale known today as just intonation gained support in the 16th century, because its principal triads, *C E G*, *F A C*, and *G B D*, had these just ratios. But an important minor triad, *D F A*, is harsh in just intonation, and other unsatisfactory triads result when this tuning is extended to the complete chromatic scale.

To correct the worst of these pitch errors, various systems of temperament were devised, in which the intonation of some (or all) notes was altered slightly, so that no intervals would be unusable. In practice, temperament was often

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haphazard, a rule of thumb which some musical theorists exalted into definite systems. More important are the regular systems, those in which a particular value of the perfect fifth (as $C\,G$) is used as the tuning unit. The meantone temperament is a regular system in which the fifth (3/2) is flatted by 1/4 syntonic comma, and the major third is pure (5/4). A comma is a very small interval, the difference between two larger intervals. The syntonic, or Ptolemaic, comma is the difference between two tones, $(9/8)^2$, and a pure major third, (5/4). Since the difference between intervals is expressed by the quotient of their ratios, this comma has the ratio of 81/64 to 5/4, or 81:80. A fifth tempered by 1/4 syntonic comma then has the ratio $(3/2)(80/81)^{1/4}=5^{1/4}:1$, in place of 3:2. Other regular systems in which the fifth is tempered by an aliquot part of the comma, such as 1/3 or 1/5, may be thought of as varieties of the meantone temperament.

Today the only tuning system in common use is equal temperament, in which the fifth is tempered by 1/12 ditonic comma. The ditonic, or Pythagorean, comma is the difference between six tones, $(9/8)^6$, and the pure octave, (2/1); that is, the ratio 531441:524288, which lies between 75:74 and 74:73. A fifth tempered by 1/12 ditonic comma has the ratio $(3/2)(8/9)^{1/12}2^{1/12}=2^{7/12}:1$. This same ratio is obtained directly from the consideration that the interval of the fifth contains 7 semitones; that is, it is 7/12 of an octave.

Equal temperament was used on fretted instruments (lutes and viols) in the early 16th century. It was advocated for keyboard instruments before the end of the 16th century, but was not universally accepted until the middle of the 19th century. The reason for its slow acceptance was the sharpness of its major thirds. These deviations will be clearer if the intervals are expressed in a logarithmic unit known as the cent, $2^{11/200}$. The octave contains 1200 cents; the fifth of equal temperament, 700; its third, 400. The pure fifth, with ratio 3:2, contains 1200 (log 3/2) $\div$ log 2 = 702 cents, and thus has been slightly flatted in equal temperament; but the pure third, with ratio 5:4, contains 386.3 cents, and thus has been sharped by almost 14 cents. The syntonic comma contains 21.5 cents; the ditonic comma, 23.5.

The tremendous advantage of equal temperament over other regular systems of 12 notes is that it is a closed, or cyclic, system. In a cyclic tuning system the nth power of the value chosen for the fifth will be a higher octave of the initial note. A great many suggestions have been made from time to time for improving the quality of the thirds by increasing the number of notes in the octave. The best of these systems operate upon the principle that an equal division of the octave is much to be preferred to an extension of just intonation. In making the division, care must be taken that the improvement of the thirds does not result in too great an impairment of the fifths. An increase in the number of notes in the octave has the further advantage that such enharmonic pairs as $G\sharp$ and $A\flat$ can be differentiated.

2. Systems of multiple division. In the middle of the 16th century Nicola

Vicentino described his six-manual Archicembalo, in which the octave is divided into 31 equal parts. Since he wrote before the invention of logarithms, he had no way of expressing these intervals by numbers on a monochord. But he directed that the fifths are to be tempered "according to the usage and tuning common to all the keyboard instruments," that is, according to the 1/4 comma meantone temperament. It remained for Christian Huyghens to show that the two systems are almost identical. The meantone fifth contains 696.6 cents; its third 386.3. The fifth of the 31-division contains 696.8 cents; its third, 387.1.

The division of the octave into 19 equal parts was accomplished a few years after Vicentino's successful attempt at multiple division. Both Zarlino and Salinas discussed harpsichords with 19 keys to the octave, and the latter advocated a system in which the fifth is tempered by 1/3 comma. This corresponds very closely to the equal 19-division. The fifths of both systems contain 694.7 cents; their thirds, 378.9. The large distortion of both intervals would make this system inferior to the ordinary 1/4 comma meantone temperament were it not that it is a cyclic system of only a few more notes than our present equal temperament of 12 notes. There have been eloquent advocates of the 19-division even in our own day, including the Russian-American, Joseph Yasser.

The best cyclic division with fewer than 100 parts is the 53-division. It was implied by the Pythagoreans, who held that there are 4 commas in the diatonic semitone and 9 in a tone, whence the octave will contain $5 \cdot 9 + 2 \cdot 4 = 53$ commas. Since $1200 \div 53 = 22.6$ cents, this is a mean comma between the syntonic and ditonic commas. It was appreciated by a medieval Chinese theorist, King Fâng, and was referred to by 17th century European musical theorists (Mersenne and Kircher); it was used by Nicholas Mercator as a "common measure" for all intervals, and applied by Bosanquet to the "generalized" keyboard of his Enharmonic Harmonium. The fifths of the 53-division are practically perfect (701.9 cents), and its thirds are slightly flat (384.9 cents). Since this is a "positive" system, with fifths larger than those of equal temperament, its third must be formed as a diminished fourth, as C $F\flat$. (The "negative" definition of the third in terms of fifth and octave is: $T = 4F - 2O$; the "positive," $T = 5O - 8F$.) This would be a confusing feature to a performer used to thirds that are really thirds.

Of the three systems [1] of multiple division discussed above, only the 19-division would have any practical value. The others would have been too expensive to construct and too cumbersome to play. And so, as we discuss various theories of the division of the octave, it must be realized that such a scheme is useless to the musician. Of course, some of the workers in this field have tried seriously to perfect instruments for multiple division, believing they would help the cause of music thereby. Events have proved them wrong. But others, who glibly talked of dividing the octave into more than 100 parts, had nothing to say about how their theories could be applied to instruments. They may have been deceiving themselves, but probably were aware that this was nothing but mathematical speculation.

3. Theories of multiple division. Joseph Sauveur, the French geometer and acoustician, was greatly interested in multiple division. In one of his articles [2] he gave a very impressive list of octave divisions, consisting of 25 terms: 12, 17, 19, 31, 43, 50, 53, 55, 67, 74, 98, 105, 112, 117, 122, 136, 141, 153, 160, 177, 184, 189, 208, 232, 256. All of these except 12, 17 (which he calls an Oriental system), and 53 are what Bosanquet called "negative," that is, they have fifths that are flatter than those of equal temperament, and thus form their major thirds properly.

Sauveur's theory is based on the division of the tone, as $C\ D$, into a diatonic semitone, as $C\ D\flat$, and a chromatic semitone, as $D\flat\ D$. In just intonation, if the ratio 16:15 or 112 cents is taken for the diatonic semitone, and the ratio 135:128 or 92 cents for the chromatic semitone, the ratio of the diatonic to the chromatic semitone will be 112:92 or 28:23. Convenient approximations to this ratio are 5:4, 6:5, and 11:9. Since the octave contains 7 diatonic and 5 chromatic semitones, we may use the formula, $O=7d+5c$, where $d>c$. If $d/c=5/4$, $O=7\cdot 5+5\cdot 4=55$ parts; if $d/c=6/5$, $O=7\cdot 6+5\cdot 5=67$ parts; if $d/c=11/9$, $O=7\cdot 11+5\cdot 9=122$ parts.

In Sauveur's list there are combined several independent series, each determined by the formula $d=c+i$, where $i=1, 2, 3, 4$. If, for example, $i=4$, $O=7(c+4)+5c=12c+28$. Using only odd values of c from 7 through 19, Sauveur obtained 112, 136, 160, 184, 208, 232, 256. The ratio of the third to the fifth to the octave in this series is $(4c+8):(14c-5):(24c-8)$. The limit of these ratios is 4:7:12, as in equal temperament. Therefore, as c increases beyond a certain value, the fifths continue to improve, but the thirds become sharper and sharper, since equal temperament has good fifths and very sharp thirds. This will be true of each of Sauveur's series. It would seem to be a false theory by which series are generated in which the middle terms are the best. Romieu [3] and Bosanquet [4], both of whom did outstanding work in this field, had no better theories than Sauveur.

M. W. Drobisch [5] made the first notable contribution to the theory of multiple division of the octave through the use of continued fractions. Having expressed the ratio of the fifth to the octave (log 3/2:log 2) as a decimal, 0.5849625, or as a fraction, 46797/80000, he used ordinary continued fractions to find successive approximations to this ratio. He obtained for his denominators (the octaves) this series: 2, 5, 12, 41, 53, 306, 665, [15601], $\cdots$. Then, apparently not trusting these results, he found all the powers of 3/2 from the 13th to the 53rd, to ascertain which of them approach a pure octave. This should have checked closely with his previous list, to which 17 and 29 would be semiconvergents. His complete list is: 17, 19, 22, 29, 31, 41, 43, 46, 51, 53. Having eliminated all positive divisions (those with raised fifths), he still had 19, 31, and 43 to add to his previous list.

Drobisch's continued fraction expansion was the first really scientific method of dividing the octave with regard to the principal consonances, the thirds and the fifths. The difficulty with it is that there are three magnitudes to be com-

pared (third, fifth, octave), but only one ratio (third to octave, or fifth to octave) can be approximated by binary continued fractions. If we must choose a single ratio, it is better to use the ratio of fifth to octave, as Drobisch did, since the third may be expressed in terms of the fifth. But the "negative" formula, $T=4F-2O$, is valid only through $O=12$. The syntonic comma (81:80 or 21.5 cents) is about 1/56 octave; hence this formula will fail to give a correct number of parts for the third in any octave division greater than 28. For example, if $O=41$ and $F=24$, the formula gives $T=14$, whereas the correct value is 13. Knowing the approximate size of the comma, we can correct the formula to read: $T=4F-2O-[O/56]$. But even this would give only by accident a value for the third with as small an error as that for the fifth in the same division.

4. Use of ternary continued fractions. The desired solution can be obtained only by ternary continued fractions, which are a means by which the ratios of three numbers may be approximated simultaneously, just as the ratios of two numbers may be approximated by binary continued fractions. C. G. J. Jacobi [6] was the first person to explore the possibilities of ternary continued fractions. The most thorough study of the subject has been made by Oskar Perron in half a dozen articles over a period of 30 years [7]. American workers in this field include D. H. Lehmer, J. B. Coleman, and P. H. Daus, most of their articles being published in the *American Journal of Mathematics*. The principal questions discussed in these papers are convergence and periodicity, especially the relation between periodic ternary continued fractions and the roots of cubic equations. None of the theory developed by these men appears to have any bearing upon the present problem.

Jacobi's algorithm. P. H. Daus has stated Jacobi's algorithm for ternary continued fractions as follows [8]:

If u_1, v_1, w_1 be any three numbers, we define a ternary continued fraction expansion for them by the equations

$$u_{n+1} = v_n - p_n u_n; \qquad v_{n+1} = w_n - q_n u_n; \qquad w_{n+1} = u_n. \tag{1}$$

. . . . . .

The numbers (A, B, C), defined by the recursion formulas (3) below, form the nth convergent set to the ternary continued fraction

$$\left(\frac{v_1}{u_1}, \frac{w_1}{u_1}\right) = (p_1, q_1; p_2, q_2; \cdots; p_n, q_n; \cdots).$$

$$\begin{aligned} A_n &= q_n A_{n-1} + p_n A_{n-2} + A_{n-3}, \\ B_n &= q_n B_{n-1} + p_n B_{n-2} + B_{n-3}, \\ C_n &= q_n C_{n-1} + p_n C_{n-2} + C_{n-3}, \end{aligned} \tag{3}$$

with the initial conditions

$$\begin{pmatrix} A_{-2} & A_{-1} & A_0 \\ B_{-2} & B_{-1} & B_0 \\ C_{-2} & C_{-1} & C_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In his earlier article [9], Daus had shown that

$$\begin{pmatrix} A_{n-2} & A_{n-1} & A_n \\ B_{n-2} & B_{n-1} & B_n \\ C_{n-2} & C_{n-1} & C_n \end{pmatrix} = \prod_1^n \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & p_n \\ 0 & 1 & q_n \end{pmatrix},$$

from which the formulas in (3) follow. In [9], Daus, following Jacobi's erroneous example, had interchanged p's and q's in the formulas for A_n, B_n, and C_n, given correctly in (3).

Jacobi's method in the tuning problem. For our musical problem

$$u_1 = \log 5/4; \quad v_1 = \log 3/2; \quad w_1 = \log 2.$$

The ratios $A_n : B_n : C_n$ represent successive approximations to the ratios of these three logarithms. The musical interpretation is that if the octave is divided into C_n parts, A_n parts are a good approximation to the major third, B_n to the perfect fifth. In Table I are shown the results of applying Jacobi's expansion to the three logarithms.

TABLE I. JACOBI'S TERNARY CONTINUED FRACTIONS, WITH APPLICATION TO THE TUNING PROBLEM

p_n	q_n	A_n	B_n	C_n
1	3	1	1	3
0	1	1	2	3
1	7	8	15	25
0	1	9	16	28
0	1	10	18	31
0	2	28	51	87
3	8	478	263	817

There are two serious faults with these results. In the first place, the expansion converges too rapidly, and we are interested chiefly in small values, those for which $C<100$. In the second place, the first few terms are foreign to every other proposed solution, such as those by Sauveur and Drobisch on previous pages. Compare this table with Table IV that follows, which is a composite of Tables I–III.

Reversed Expansion. Jacobi's method is not the only way to obtain a ternary continued fraction expansion. What might be called the reverse of Jacobi's expansion is defined as follows:

$$u_{n+1} = u_n - p_n v_n; \quad v_{n+1} = w_n - q_n v_n; \quad w_{n+1} = v_n.$$

Here

$$\begin{pmatrix} A_{n-2} & A_{n-1} & A_n \\ B_{n-2} & B_{n-1} & B_n \\ C_{n-2} & C_{n-1} & C_n \end{pmatrix} = \prod_{1}^{n} \begin{pmatrix} 1 & 0 & p_n \\ 0 & 0 & 1 \\ 0 & 1 & q_n \end{pmatrix}$$

from which

$$A_n = q_n A_{n-1} + A_{n-2} + p_n,$$
$$B_n = q_n B_{n-1} + B_{n-2},$$
$$C_n = q_n C_{n-1} + C_{n-2},$$

with the same initial conditions as before.

TABLE II. REVERSED TERNARY CONTINUED FRACTIONS, WITH APPLICATION TO THE TUNING PROBLEM

p_n	q_n	A_n	B_n	C_n
0	1	0	1	1
0	1	0	1	2
1	2	1	3	5
2	2	4	7	12
0	3	13	24	41
0	1	17	31	53
0	5	98	179	306
1	2	214	389	665

Since v_1 is log 3/2 and w_1 is log 2, and since B_n and C_n are independent of p_n, the series for the ratio of perfect fifth to octave (B_n:C_n) will be the same as that obtained by Drobisch by binary continued fractions. However, the correction for the major third for octave divisions greater than 28 is no longer needed, since the correction is made directly by the formula for A_n, with its extra member.

It probably is an accident that in this particular problem the solution by the reversed method yields a more familiar series than Jacobi's expansion does. (Again compare with Table IV.) Its rate of convergence is slightly lower, but it is still somewhat high. The method is presented here chiefly as an added basis for the expansion next to be discussed.

Defects of both expansions. Since neither the Jacobi nor the reversed ternary continued fraction expansion converges slowly, some method must be devised for slower convergence. In binary continued fractions, about half of the semi-convergent sets that lie between two convergent sets form better approximations to the ratio of the given numbers than the former of the two convergent sets does. But semi-convergents are a doubtful expedient in ternary continued fractions. Although their peculiarities have never been completely charted, a brief study of them by the present writer shows that a careful selection of coefficients provides semi-convergents superior to the *latter* of the two convergent sets, something impossible in binary continued fractions. For example, between

the second and third sets of Table I may be inserted the semi-convergent sets (taking q_3 as 5 and 6, respectively) 6 11 19 and 7 13 22, both of which, as can be seen from Table IV, are superior to the next convergent set of Table I, 8 15 25.

Mixed expansion. It is possible, however, to obtain fairly slow convergence without recourse to semi-convergents. At any stage of the expansion there are two possible divisors, u_n and v_n, the former being used in the Jacobi expansion and the latter in the reversed method. If either of these expansions is used, the rate of convergence is likely to be somewhat irregular, but approaches a mean value. But if fast convergence is desired, one should consistently divide by whichever of u_n and v_n is the smaller. In our tuning problem, slow convergence is the desideratum. Hence one should always divide by the larger of u_n and v_n.

This mixed expansion, as it may be called, presents certain difficulties, since it is no longer possible to express an approximation in terms of the three immediately preceding approximations. No new definitions of u_{n+1}, v_{n+1}, and w_{n+1} are needed; for at any stage either the Jacobi (J) or the reversed (R) expansion is to be used.

Then, in terms of a matrix product, for a J stage,

$$\begin{pmatrix} A_{n-2} & A_{n-1} & A_n \\ B_{n-2} & B_{n-1} & B_n \\ C_{n-2} & C_{n-1} & C_n \end{pmatrix} = \begin{pmatrix} A_{n-k-3} & A_{n-2} & A_{n-1} \\ B_{n-k-3} & B_{n-2} & B_{n-1} \\ C_{n-k-3} & C_{n-2} & C_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & p_n \\ 0 & 1 & q_n \end{pmatrix},$$

where k is the number of R stages between this stage and the last previous J stage. For an R stage, substitute on the right the matrix for the reversed expansion,

$$\begin{pmatrix} 1 & 0 & p_n \\ 0 & 0 & 1 \\ 0 & 1 & q_n \end{pmatrix}.$$

From the above we obtain,

for an R stage: $\quad S_n = q_n S_{n-1} + S_{n-2} + p_n S_{n-k-3}$;

for a J stage: $\quad S_n = q_n S_{n-1} + p_n S_{n-2} + S_{n-k-3}$.

It is understood that

$$S_n = A_n,\ B_n, \quad \text{or} \quad C_n.$$

5. Mixed expansion in the tuning problem. In the application of the mixed method to our tuning problem, we shall use the Jacobi (J) method whenever $u_n > v_n$, and the reversed (R) method whenever $u_n < v_n$. Hence all the p's will be 0. In Table III this application is shown, together with a determinant in which A_n, B_n, and C_n are the coefficients of a, b, and c, respectively, in the expansion of the determinant.

TABLE III. MIXED TERNARY CONTINUED FRACTIONS, WITH APPLICATION TO THE TUNING PROBLEM

$$\begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & p_1 & q_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_2 & 1 & q_2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & p_3 & q_3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & p_4 & q_4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_5 & 1 & q_5 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & p_6 & q_6 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_7 & 1 & q_7 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_8 & 0 & 1 & q_8 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_9 & 0 & 0 & 1 & q_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & p_{10} \end{vmatrix} \quad \begin{vmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ q_{10} \end{vmatrix}$$

	p_n	q_n	A_n	B_n	C_n
	0	1	0	1	1
R	0	1	0	1	2
J	0	1	1	1	2
J	0	1	1	2	3
R	0	1	2	3	5
J	0	1	2	4	7
R	0	1	4	7	12
R	0	1	6	11	19
R	0	1	10	18	31
J	0	1	11	20	34
J	0	1	17	31	53
R	0	1	28	51	87
J	0	1	38	69	118
R	0	4	180	327	559
J	0	1	197	358	612

Table IV contains all the terms from the other three tables, through $C=118$, together with a few other good approximations not found in any of the previous tables. The errors in cents for the major third (in some cases to 0.1 cent) have been computed by the formula: Error $A_n=386.3-1200A_n/C_n$; the errors for the fifth by: Error $B_n=702-1200B_n/C_n$. The total error is taken as $|\text{Error } A_n| + |\text{Error } B_n|$.

TABLE IV. COMPOSITE OF APPROXIMATIONS TO TUNING RATIOS (LOG 5/4: LOG 3/2: LOG 2), WITH BEST APPROXIMATIONS STARRED

A_n	B_n	C_n	Error A_n	Error B_n	Total Error
* 0	1	1	−386	498	884
* 0	1	2	−386	−102	488
* 1	1	2	214	−102	316
* 1	1	3	14	−302	316
* 1	2	3	14	98	112
* 2	3	5	94	18	112
1	3	5	−146	18	164
* 2	4	7	− 43	− 16	59
* 3	5	9	14	− 35	49
* 3	6	10	− 26	18	44
* 4	7	12	14	− 2	16
* 6	11	19	− 7	− 7	14
* 7	13	22	− 4	7	11
8	15	25	− 2	18	20
9	16	28	− 1	− 16	17
*10	18	31	.8	− 5.2	6.0
*11	20	34	1.9	3.9	5.8
13	24	31	− 6	1	7
*17	31	53	− 1.4	− .1	1.5
*28	51	87	− .1	1.4	1.5
*38	69	118	.2	− .5	.7

Of the starred best approximations in Table IV, only $C=9$, 10, and 22 are missing from Table III; but none of these occurs in Tables I or II either. The only serious omission from Table III is the Hindoo division, $C=22$, and it would occur as a semi-convergent, by assuming a J instead of an R for the term where 31 now stands. Note the two terms, 31 and 34, nearly the same size: it would have been impossible to include them both (even if one had been a semi-convergent) if binary or Jacobi ternary continued fractions had been used. Both 53 and 118 are present; but between them lies 87, a term superior to 65, the middle term in the binary series. The last term shown, 612, was said by Bosanquet [4] to have been considered excellent by Capt. J. Herschel; it would have had no place in any other series derived by continued fractions in which 559 was also present.

6. Euler's problem. This mixed method for ternary continued fraction expansions has been shown to satisfy the need for slow convergence in our special problem. It should be equally valuable wherever slow convergence is desired. In this connection, let us return to Jacobi. His article on ternary continued fractions was preceded, both literally and logically, by an article [10] in which he treated integral solutions of a linear equation in any number of variables. He made special reference, however, to Euler's problem: to solve in integers the equation $Aa+Bb+Cc=0$, where the capital letters are given integers, possibly integral approximations to irrational numbers.

Euler's first example was $49a+59b+75c=0$. To obtain a general solution, he placed u on the right, and then replaced it by 0 for the particular solution. His method was to divide two coefficients by the remaining coefficient, always dividing by the *smaller* remainder in each stage. Thus his method made for rapid convergence. It is irregular, but for a purpose, as is our method of dividing by the larger remainder. In the example just given, the coefficients are so chosen that the smaller remainder is always on the left, thus agreeing with Jacobi's method, which he first used in this connection before applying it to continued fractions. Both Euler and Jacobi obtained as the smallest solution: $a=-13$, $b=-7$, $c=14$.

Euler's second example was $1{,}000{,}000a+1{,}414{,}214b+1{,}732{,}051c=u$, where the second and third coefficients will be recognized as approximations to $\sqrt{2}$ and $\sqrt{3}$, with the decimal point omitted. Euler again divided by the smaller remainders, and thus got a different result from Jacobi, who divided by the left-hand remainders. With $u=0$, Euler's smallest solution was: $a=8104$, $b=-6889$, $c=946$; Jacobi's, $a=282$, $b=-2377$, $c=1778$.

7. Mixed expansion in Euler's problem. The results already obtained in this paper suggest that Jacobi's results may have been better than Euler's in this second problem, not because his method was regular, but because of the slower convergence arising from a casual mixture of small and large remainders as divisors. If, then, the larger remainders are used uniformly as divisors, as in the tuning problem, better results than Jacobi's might be expected. As applied

to Euler's first problem, our mixed method gives: $a=u-8i+j$, $b=3u+13i-11j$, $c=-3u-5i+8j$. If $u=0$, $i=1$, $j=1$, then $a=-7$, $b=2$, $c=3$. The Euler-Jacobi solution above is then obtained by letting $i=2$, $j=3$.

Again, in Euler's second problem, division by the larger remainders yields superior results. This method gives: $a=51u-2403\alpha+895\beta$, $b=264u+1829\alpha-1402\beta$, $c=-245u-106\alpha+628\beta$. If $u=0$, $\alpha=1$, $\beta=1$, then $a=-1508$, $b=427$, $c=522$. Euler's solution is obtained by letting $\alpha=-3$, $\beta=1$; Jacobi's, with $\alpha=1$, $\beta=3$.

8. Conclusion. Insufficient study has been given the mixed expansion for ternary continued fractions to ascertain all of its possibilities and weaknesses. Without the use of semi-convergents, the mixed method clearly provides slower convergence than Jacobi's method. It can be used to good advantage also when fast convergence is desired, by a simple reversal of the hypotheses. In this case the divisors are so selected than none of the p's will be 0. Thus the mixed expansion would seem to fulfill a definite need which Jacobi's expansion does not meet.

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CLAIRAUT AND THE ORIGIN OF THE DISTANCE FORMULA

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The familiar distance formula of plane analytic geometry

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

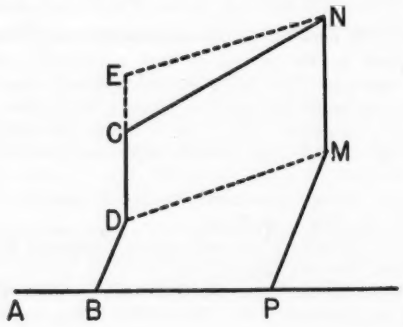
is ascribed [1] universally to Lacroix, who published it [2] in 1797 and 1798. Paradoxically, the corresponding formula for space of three dimensions, once ascribed to Monge [3], has been traced back to Clairaut's classical *Recherches sur les courbes à double courbure* of 1731. Coolidge consequently expressed the traditional view when he wrote:

> Clairaut published the distance formula in three dimensions three-quarters of a century before LaCroix gave the corresponding formula in plano.

Coolidge prudently added, however,

> My conclusion is that LaCroix's formula had been for a long time more or less common property and could be found by sufficiently diligent research [4].

It is the object of this brief note to point out that Coolidge was quite correct in his conclusion. The distance formula for two dimensions can indeed be found before LaCroix, and exactly where one would expect to find it—in Clairaut!



The *Recherches* of Clairaut, composed when he was only sixteen years old, is devoted primarily to twisted curves in three-space, and hence the formula for distance in a plane appears but incidentally, as a preliminary to the determination of the equation of the sphere. Let the center of the sphere be C with coördinates $AB = \pm a$, $BD = \pm b$, and $DC = \pm c$, with respect to the axis AB and the reference plane ABD; and let N be any point on the sphere with coördinates $AP = x$, $PM = y$, and $MN = z$. Then Clairaut wrote [5] that $EN = MD$

$=\sqrt{\overline{x\mp a}^2+\overline{y\mp b}^2}$, possibly the first time that the distance formula had appeared in print. He gave the analogue for three-space immediately afterwards as $f=\sqrt{\overline{x\mp a}^2+\overline{y\mp b}^2+\overline{z\mp c}^2}$, where $f=CN$. Consequently, to Clairaut, pending further evidence, must go the credit for the distance formula, both in two dimensions and in three, for his forms differ but slightly from the modern equivalents, mainly in the failure to regard the literal constant quantities a, b, and c as indifferently either positive or negative. More modern forms, also overlooked by historians, were given in 1773 by Lagrange for both two and three dimensions. In the preface to Clairaut's *Recherches* there is a hint of a system of spherical coördinates, but Lagrange was one of the first to develop this; and it is in his transformation from Cartesian to spherical coördinates that one finds [6] distances given by the expressions $\sqrt{(x-a)^2+(y-b)^2}$ and $\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$.

The contribution of Clairaut in this connection should not, however, be exaggerated. The distance formulas are, after all, obvious analytical expressions of an ancient theorem, named for Pythagoras, but known to the Babylonians of some four thousand years ago. There can be little doubt but that their equivalents were familiar to the earliest analytical geometers, including the inventors, Fermat and Descartes. The equations of circles and spheres, given long before Clairaut, are tantamount to distance formulas; and the rectification of curves, known since 1659, is dependent upon some such equivalent. The formulas for distance in infinitesimal analysis, $\sqrt{dx^2+dy^2}$ and $\sqrt{dx^2+dy^2+dz^2}$, also appear in Clairaut's *Recherches*; but these are not attributed to him. It would not be surprising if "sufficiently diligent research" would reveal explicit, as well as implicit, anticipations of Clairaut's distance formulas, not only for the calculus, but also for analytic geometry.

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MATHEMATICAL NOTES

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DETERMINANTS OF FOURTH ORDER MAGIC SQUARES

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1. Introduction. In the editorial note to problem E 674 [this MONTHLY, vol. 53, 1946, p. 99] the question was raised, "What is the minimum necessary condition that the determinant of a magic square be zero?" The conditions are given below for fourth order squares whose elements are in arithmetic progression.

Dudeney, in his "Amusements in Mathematics" (1917), page 120, classifies fourth order magic squares into twelve types according to the relative positions of the conjugate pairs whose sum, X, is one-half the constant. These twelve types fall into four "determinant" types. As indicated on the opposite page, Types I, II, III are included in Determinant Type A; Types IV, V, VI are in Determinant Type B; Types VII, VIII, IX, X are in Determinant Type C; and Types XI, XII are in Determinant Type D. Type I is a Nasik (pandiagonal) square; Types II, III, IV and V are semi-Nasik squares; Type VI may be semi-Nasik or simple; and Types VII, VIII, IX, X, XI and XII are simple magic squares. The type numbers and patterns (in which connected cells contain conjugate pairs) are Dudeney's.

2. Transitions. The transitions within the "determinant" types are as follows:

Type A.

Type II. Interchange 2nd and 3rd rows, and then interchange 2nd and 3rd columns to get Type I.

Type III. Interchange 3rd and 4th columns, and then interchange 3rd and 4th rows to get Type I.

Type B.

Type IV. Place first row below the 4th row to get Type VI.

Type V. Interchange 3rd and 4th rows to get Type VI.

Type C.

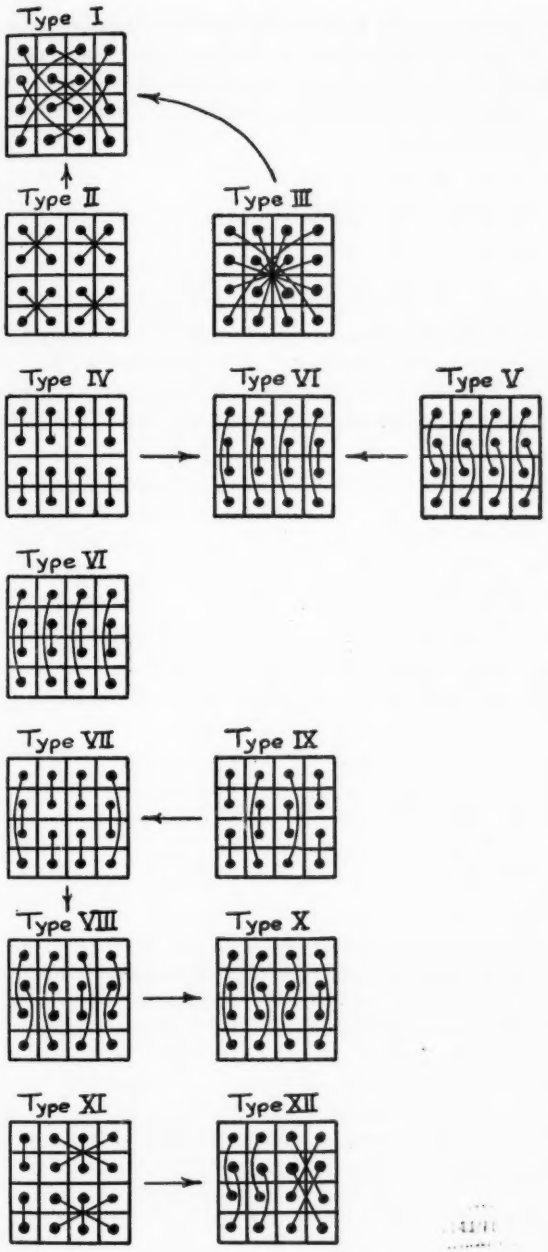
Type VII. Interchange 3rd and 4th rows, and then place first row below the 4th row to get Type VIII.

Type VIII. Interchange first and 2nd rows to get Type X.

Type IX. Place first row below 4th row to get Type VII.

Type D.

Type XI. Interchange 2nd and 3rd columns and then 2nd and 3rd rows to get Type XII.



Type I

1	14	7	12
15	4	9	6
10	5	16	3
8	11	2	13

Type II

1	7	14	12
10	16	5	3
15	9	4	6
8	2	11	13

A

Type VI

1	14	12	7
4	15	9	6
13	2	8	11
16	3	5	10

Type VI

1	12	14	7
4	15	9	6
13	2	8	11
16	5	3	10

B

Type VII

9	16	2	7
12	1	15	6
5	4	14	11
8	13	3	10

Type VIII

16	4	5	9
14	2	7	11
1	15	10	8
3	13	12	6

C

Type XI

15	6	9	4
2	13	8	11
7	12	1	14
10	3	16	5

D

3. Evaluations. Since the absolute value, Δ, of a determinant is unaltered by transposition of columns or of rows, by reflection or by rotation, it will be necessary to evaluate only one of each determinant type, provided that no property of the diagonals is used in evaluating the representative chosen. The general square may be represented conveniently by

$$\begin{array}{cccc} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & P & Q \end{array}$$

and the conjugate sum by X so that the constant is $2X$. Evidently the sum of each of the columns, rows and principal diagonals in all of the squares is $2X$.

Type A. Schell showed in E 674 that $\Delta=0$ for Type I. All members of Type A therefore have $\Delta=0$. This may be shown also by evaluating Type II, which may be written

$$\begin{vmatrix} A & B & C & D \\ X-B & X-A & X-D & X-C \\ I & J & K & L \\ X-J & X-I & X-L & X-K \end{vmatrix}.$$

By subtracting the first column from the 2nd and the 3rd column from the 4th, *then* subtracting the 2nd row from the first and the 4th row from the 3rd, *then* adding the first and 3rd columns (noting that $A+B+C+D=2X=I+J+K+L$),

$$\begin{vmatrix} A+B-X & 0 & 0 & 0 \\ X-B & B-A & 2X-B-D & D-C \\ I+J-X & 0 & 0 & 0 \\ X-J & J-I & 2X-J-L & L-K \end{vmatrix} = 0$$

is secured.

Type B. It is evident upon inspection that if the first row of Type IV be added to the second row, and the third row be added to the fourth, a determinant is secured with two rows of identical elements, namely X. Therefore, for Type IV and for all of Type B, we have $\Delta=0$.

Types C and D. Type VII may be written

$$\begin{vmatrix} A & B & C & D \\ E & X-B & X-C & H \\ X-E & J & K & X-H \\ X-A & X-J & X-K & X-D \end{vmatrix}.$$

From the second and third rows, $2X-B-C+E+H=2X$ and $2X-E-H+J+K=2X$, so $B+C=E+H=J+K$. The determinant is evaluated by adding the first three columns to the fourth, then the first three rows to the fourth, and factoring out $4X$. Then adding the first row to and subtracting the fourth row from the second row, we obtain

$$\Delta = -4X(A+E-X)\begin{vmatrix} B & C & 1 \\ J & K & 1 \\ X & X & 2 \end{vmatrix}.$$

The evaluation is completed by adding the first column to the second, and then subtracting the first row from the second. Thus

$$\begin{aligned}\Delta &= -8X[A-(X-E)][B-J][C-(X-B)] \\ &= -8X(A-I)(B-J)(C-F).\end{aligned}$$

Therefore, if all elements are distinct, we have $\Delta \neq 0$ for Type VII and for the other members of Type C.

The other members of Type C may be evaluated by expansion, or by transposing Type VII and setting the square thus secured into a one-to-one correspondence with the standard square. Types XI and XII are separately evaluated. Thus for the members of Types C and D:

Type VII	$\Delta = -8X(A-I)(B-J)(C-F)$
Type VIII	$\Delta = -8X(A-E)(B-F)(C-N)$
Type IX	$\Delta = -8X(A-I)(B-J)(C-N)$
Type X	$\Delta = 8X(A-E)(B-F)(C-J)$
Type XI	$\Delta = 8X(A-I)(B-D)(C-A)$
Type XII	$\Delta = 8X(A-E)(B-A)(C-D).$

4. Conclusion. It is therefore apparent that in order for a fourth order magic square with elements in arithmetic progression to have a zero determinant, it is necessary and sufficient that it meet the conditions for a semi-Nasik square or that it be of Type VI.

Since the sum S of all the elements of the square is $8X$, a secondary result is that if all the elements are integers then Δ/S is an integer. This property is proven in another manner in the solution to problem E 813, where it is also shown that Δ/S is an integer for squares of order n when n is odd and also when n is even if the difference d is even.

THE NUMBER OF DISTANCES IN A CUBICAL NETWORK

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We consider the set of points $\{(a, b, c)\}$ in a three dimensional Euclidean space, where each coordinate a, b and c assumes the values $0, 1, \cdots, n$ independently of the values of the other coordinates. Let A_k be the number of straight lines of length k which connect one point of the set with another, and let A_0 equal the total number of points in the set. Then the following identity holds:

$$A_0 + 2\sum_{d\geqq 1} A_d x^{d^2} = \left[(n+1) + 2\sum_{\nu=1}^{n}(n-\nu+1)x^{\nu^2}\right]^3. \tag{1}$$

Proof: Consider the six fold sum

$$\sum_{\alpha=0}^{n}\sum_{\beta=0}^{n}\sum_{\gamma=0}^{n}\sum_{\lambda=0}^{n}\sum_{\mu=1}^{n}\sum_{\nu=0}^{n} x^{(\alpha-\lambda)^2+(\beta-\mu)^2+(\gamma-\nu)^2}. \tag{2}$$

Each term of this sum corresponds to either (1) a point of the set $\{(a, b, c)\}$ or (2) a pair of points of this set.

Case (1). For every point (a', b', c') there will be exactly one term in the sum (2) for which

$$\alpha = \lambda = a', \qquad \beta = \mu = b', \qquad \gamma = \nu = c'$$

and the value of this term is 1.

Case (2). For every pair of distinct points (a'', b'', c'') and (a''', b''', c''') there will be two terms with the exponent $(a''-a''')^2+(b''-b''')^2+(c''-c''')^2$. This exponent is the square of the distance between the points. Since every term of the sum (2) arises under case (1) or case (2), the coefficient of x^{k^2} when the terms are combined, will be equal to $2A_k$. ($A_k=0$ if there are no terms with exponent k^2.) The constant term is equal to A_0. Hence (2) is equal to the left hand side of (1).

The following transformations show that (2) is also equal to the right hand side of identity (1). We write (2) as follows:

$$\left[\sum_{\alpha=0}^{n}\sum_{\beta=0}^{n} x^{(\alpha-\beta)^2}\right]^3 \tag{3}$$

Let $\alpha-\lambda=\nu$, then ν takes on the values $-n, -n+1, \cdots, -1, 0, 1, \cdots, n$. For each value of ν, there are $n-|\nu|+1$ pairs α, λ for which $\alpha-\lambda=\nu$. Hence (3) can be written

$$\left[\sum_{\nu=-n}^{n}(n-|\nu|+1)x^{\nu^2}\right]^3, \tag{4}$$

and finally as

$$\left[(n+1)+2\sum_{1}^{n}(n-\nu+1)x^{\nu^2}\right]^3, \tag{5}$$

which is the form of the right hand side of (1).

Identities of a similar nature can be obtained for a set of points in n dimensional space, and also for points whose array is rectangular rather than cubical.

NOTE ON THE GAMMA FUNCTION

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In this note we use the inequality of Hölder to establish the classical asymptotic relation

$$\lim_{x\to\infty}\frac{\Gamma(x+a)}{x^a\Gamma(x)}=1 \tag{1}$$

for real a and x.

Proof: Let $0<a<1$, $p=1/a$, $q=p/(p-1)=1/(1-a)$, $f(t)=e^{-a^{-1}t}t^{ax}$, $g(t)=e^{-(1-a)^{-1}t}t^{(1-a)x+a-1}$,

and apply the inequality

$$\int_0^\infty f(t)g(t)dt \leqq \left\{\int_0^\infty f(t)^p dt\right\}^{1/p}\left\{\int_0^\infty g(t)^q dt\right\}^{1/q}$$

to obtain

$$\Gamma(x+a)=\int_0^\infty e^{-t}t^{x+a-1}dt \leqq \left\{\int_0^\infty e^{-t}t^x dt\right\}^a\cdot\left\{\int_0^\infty e^{-t}t^{x-1}dt\right\}^{1-a} \tag{2}$$

$$=[\Gamma(x+1)]^a[\Gamma(x)]^{1-a}.$$

Since

$$\Gamma(x+1)=x\Gamma(x), \tag{3}$$

(2) becomes

$$\Gamma(x+a)\leqq x^a\Gamma(x). \tag{4}$$

Replacing a by $1-a$ in (4) we get

$$\Gamma(x+1-a)\leqq x^{1-a}\Gamma(x), \tag{5}$$

from which we obtain

$$\Gamma(x+1)\leqq(x+a)^{1-a}\Gamma(x+a), \tag{6}$$

by substituting $x+a$ for x.

Combining (4) and (6) we get

$$\frac{x}{(x+a)^{1-a}}\Gamma(x) \leqq \Gamma(x+a) \leqq x^a\Gamma(x).$$

Therefore

$$\left(\frac{x}{x+a}\right)^{1-a} \leqq \frac{\Gamma(x+a)}{x^a\Gamma(x)} \leqq 1. \tag{7}$$

Letting x tend to infinity in (7) yields (1) for $0<a<1$. The extension to all real a is immediate on repeated application of (3).

THE NUMBER OF r-TUPLES OF PAIRS OF INTEGERS

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Our problem may be stated as follows:

Let $n>1$ be a positive integer and let $(s_1, t_1), \cdots, (s_r, t_r)$ be an r-tuple of pairs of integers satisfying

$$n \geqq s_i > t_i \geqq 1,\ s_{i+1} > s_i, t_{i+1} < t_i, \qquad (i = 1, \cdots, r). \tag{1}$$

If $S(k, m)$ is the number of such r-tuples beginning with (k, m), and K is the total number of r-tuples, show that

$$S(k, m) = \binom{n-k+m-1}{n-k},$$

and hence that $K=2^{n-1}-1$.

(For example, if $n=5$, the 15 possible r-tuples are (5, 4); (5, 3); (5, 2); (5,1); (4, 3); (4, 3), (5, 2); (4, 3), (5, 1); (4, 2); (4, 2), (5, 1); (4, 1); (3, 2); (3, 2), (4, 1); (3, 2), (5, 1); (3, 1); and (2, 1).)

We first show that

$$S(k, m) = \sum_{i=k}^{n} S(i, m-1), \quad S(k, 1) = 1, \qquad (m > 1). \tag{2}$$

Indeed (2) follows from the fact that the r-tuples $(k_1, m_1), (k_2, m_2), \cdots$, with $m_2>m_1-1$, may be put into $1-1$ correspondence with the r-tuples with first term (k_1, m_1-1) via

$$(k_1, m_1), (k_2, m_2), (k_3, m_3), \cdots \leftrightarrow (k_1, m_1-1), (k_2, m_2), (k_3, m_3), \cdots;$$

while the r-tuples $(k_1, m_1), (k_2, m_1-1), (k_3, m_3), \cdots$, may be put into $1-1$ correspondence with the r-tuples beginning with (k_2, m_1-1) via

$$(k_1, m_1), (k_2, m_1-1), (k_3, m_3), \cdots \leftrightarrow (k_2, m_1-1), (k_3, m_3), \cdots.$$

To show then that

$$S(k, m) = \binom{n-k+m-1}{n-k} = \binom{n-k+m-1}{m-1}, \tag{3}$$

we first observe that (3) holds for $m=1$, since

$$S(k, 1) = \binom{n-k}{n-k} = 1$$

for all admissible k (i.e., for $n \geqq k > 1$). Likewise, for $m=2$, the r-tuples beginning with $(k, 2)$ are enumerated by remarking that there is one r-tuple consisting of $(k, 2)$ alone while the choices for a second term are $(k+1, 1), \cdots, (n, 1)$. Thus

$$S(k, 2) = n - k + 1 = \binom{n-k+1}{1}.$$

Assume, then, that (3) holds for $S(k, i)$, for $i<m$ and for any admissible k. Then (2) implies

$$S(k, m) = \sum_{i=k}^{n} \binom{n-i+m-2}{m-2} \tag{4}$$

under our induction assumption that $m>2$, so that the proof of (3) reduces to a proof of

$$\sum_{i=k}^{n} \binom{n-i+m-2}{m-2} = \binom{n-k+m-1}{m-1}, \qquad (m > 2). \tag{5}$$

With $s=m-2$, $p=n-k+s$, $j=n-i+m-2$, (5) becomes

$$\sum_{j=s}^{p} \binom{j}{s} = \binom{p+1}{s+1}. \tag{6}$$

Now (6) holds for $p=s$ or $s+1$ for any s, since

$$\binom{s}{s} = \binom{s+1}{s+1} = 1,$$

and

$$\binom{s}{s} + \binom{s+1}{s} = \binom{s+2}{s+1};$$

and (6) also holds for $p=1$ and any admissible s (i.e., here, $s=1$), since

$$\binom{1}{1} = \binom{2}{2} = 1.$$

We suppose (6) holds for the values $1, 2, \cdots, p-1$ for any admissible s (i.e., $s<p$) and re-write (6), for $p \neq s$ or $s+1$, as

$$\frac{s+1}{p-s}\binom{p}{s+1} + \sum_{j=s}^{p-1}\binom{j}{s} = \binom{p+1}{s+1}.$$

But

$$\binom{p-1}{s} + \cdots + \binom{s}{s} = \binom{p}{s+1}$$

by our induction hypothesis for $s<p-1$; and since

$$\frac{s+1}{p-s}\binom{p}{s+1} + \binom{p}{s+1} = \frac{p+1}{p-s}\binom{p}{s+1} = \binom{p+1}{s+1},$$

it follows that (6), and hence (5) and (3), are proved.

Then

$$\sum_{k=m+1}^{n} S(k, k-m) = \sum_{k=m+1}^{n}\binom{n-m-1}{k-m-1} \tag{7}$$

by (3). We let $n-m-1=t$, $k-m-1=i$ so that (7) becomes

$$\sum_{i=0}^{t}\binom{t}{i} = \binom{t}{0} + \sum_{i=1}^{t}\binom{t}{i} = 1 + (2^t - 1) = 2^{n-m+1}. \tag{8}$$

Finally, since it is clear that

$$K = \sum_{m=1}^{n-1}\Big(\sum_{k=m+1}^{n} S(k, k-m)\Big),$$

we have, by (7) and (8),

$$K = \sum_{m=1}^{n-1} 2^{n-m-1},$$

which is a geometric progression of $n-1$ terms with first term 2^{n-1} and common ratio $\frac{1}{2}$, so that

$$K = 2^{n-1}\left[\frac{1-(\frac{1}{2})^{n-1}}{1-(\frac{1}{2})}\right] = (2^{n-2} - 2^{-1})2 = 2^{n-1} - 1.$$

By varying the conditions (1) a number of similar problems may be stated, while extensions may be made by considering r-tuples of triples etc. The problem which has $t_{i+1}<t_i$ in (1) replaced by $t_{i+1}>t_i$ is considered in a study of the ideals of a certain type of nilpotent algebra being made by Sam Perlis and the author.

CLASSROOM NOTES

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POINTING OFF IN SLIDE RULE WORK

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For placing the decimal point in answers to problems solved on the slide rule there are available the method of the characteristic and the method of estimation. The first of these is precise although it has relatively few users. The second, although it borders on mysticism, seems entirely satisfactory in practice. In stating here a rigorous method of estimation, our purpose is to illuminate rather than to modify present procedure.

Suppose that a is the exact value of a fraction whose numerator and denominator are products of decimal numbers or roots of such numbers. To simplify language, suppose further that the slide rule gives the exact answer and that all numbers have exactly three significant digits. These assumptions in no way limit the generality of the results. Without using the characteristic we find from the slide rule merely the set of values

$$\cdots, 10^{-2}a, 10^{-1}a, a, 10a, 10^{2}a, \cdots \tag{1}$$

which contains the answer but we do not know which number of the set to identify with a.

We interpret the method of estimation to involve rounding the given factors to one digit. The extreme case of rounding which *decreases* a replaces a factor 199 in the numerator by 100 and consequently multiplies a by more than $1/2$. Similarly, the extreme case of rounding which *increases* a replaces a factor 199 in the denominator by 100 and consequently multiplies a by less than 2. If there are m factors in the fraction to be computed, rounding by decreasing and rounding by increasing give a lower bound a_1 and an upper bound a_2 for a and these bounds satisfy

$$(1/2)^m a < a_1 \leqq a_2 < 2^m a. \tag{2}$$

Once a_1, a_2 have been computed mentally, it is easy to pick out which of (1) lie on the interval $[a_1, a_2]$. If only one number of (1) lies on $[a_1, a_2]$, the identification is complete.

From (1), (2) it is seen that the method is successful for all values of the factors if and only if

$$\frac{a}{10} \leqq (1/2)^m a < 2^m a \leqq 10a.$$

Solving for m by logarithms or by trial gives $m \leqq 3$. The foregoing *method of bounds* therefore always works if no more than three factors are involved. More-

over, in any particular case of more than three factors it is apparent whether or not the method is reliable.

The usual method of estimation, however, employs only a single rounding, which we suppose is to the nearest single digit. The result b is not known to be either a lower bound or an upper bound in general. The extreme case of rounding replaces 149 by 100. Hence if n roundings decrease and p roundings increase the fraction, the factor by which a is multiplied is bounded below by $(2/3)^n$ and above by $(3/2)^p$:

$$(2/3)^n < \frac{b}{a} < (3/2)^p. \tag{3}$$

For the product $1.42 \times 1.43 \times 1.44 = 2.92$ the set (1) is

$$\cdots, .0292, .292, 2.92, 29.2, 292., \cdots \tag{4}$$

and the rounded value is 1. We might be tempted to select the value in (4) nearest 1. Since this gives .292, which is not the correct value, we see that some other principle of selection must be used for comparing (1) with the rounded values in a rigorous statement of the rule of estimation, although even a novice would intuitively point off correctly in the present case.

Let us agree to select the number in (1) so that the relative error of the estimated value b is numerically less with respect to that number than with respect to any other number in (1). To determine when this principle gives the correct answer, we find a necessary and sufficient condition that

$$\left| \frac{b}{a} - 1 \right| < \left| \frac{10^{-k}b}{a} - 1 \right| \tag{5}$$

for all integers k except 0.

Put $c = b/a$. It is found ($k \neq 0$) that $0 \leqq k(1-c)$ implies (5) and that $k(1-c) < 0 < (1-c)(1-10^{-k}c)$ contradicts (5). Hence only the cases where $k(1-c) < 0$ and $(1-c)(1-10^{-k}c) \leqq 0$ need be considered. It is thus found that in the presence of $k \neq 0$ (5) is equivalent to $0 < k[2-(1+10^{-k})c]$ and also to the two inequalities got from this by making $k = -1$, $k = 1$. Hence we have as the final condition

$$\frac{2}{11} < \frac{b}{a} < \frac{20}{11}. \tag{6}$$

Segment (3) lies in segment (6) if and only if

$$\frac{2}{11} \leqq (2/3)^n, \qquad (3/2)^p \leqq \frac{20}{11}. \tag{7}$$

Making the calculations by means of logarithms or by trial we find that for non-negative integers n, p (7) is equivalent to $n \leqq 4$, $p \leqq 1$. Hence we have

THEOREM 1. *If in rounding to the nearest single digit the fraction is decreased at most four times and increased at most once, the decimal point is correctly placed by requiring that the relative error of the rounded value be least.*

It will be noted that the principle identifies 2.92 as the answer in (4) and that the decision is not even close.

The result given is the best possible unless the method of estimation is changed. It is questionable whether this can be done without sacrificing generality or introducing complications which would offset the advantages of using the slide rule.

The limits given in the theorem can in general be exceeded with impunity because (i) the multipliers have values much nearer 1 than the extreme values which it is necessary to use in the above calculations and (ii) the increases and decreases tend to cancel. A lower bound $P(n, p)$ for the probability that the decimal point is correctly placed when $4<n$ or $1<p$ could be calculated. The results in practice, however, might well be better than this bound would indicate because the operator intuitively modifies the process so as to keep b near a. In fact, the writer has yet to have knowledge of a point misplaced by error of judgment on the part of an experienced operator.

A result which seems to cover the usual situations in the solution of triangles is given in Theorem 2, which is easily proved by the law of sines. The angle 6° can be replaced by the principal value of $\sin^{-1}.1$, but there is usually no advantage in doing so.

THEOREM 2. *If the ratio of two sides of a triangle is at least* 10, *the angle opposite the smaller side is less than* 6°.

SOME APPLICATIONS OF AITKEN'S METHOD OF INTERPOLATION

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Aitken's method of interpolation published in 1930 [1] seems relatively unknown, although it has been discussed by various authors [2], [3], [4], [5], [6], [8] in some detail. With the rapid increase of interest in computational techniques it should be one of the tools familiar to every student of mathematics. We shall point out some applications of the method to the mathematics of finance. We have used it for some years in those classes with considerable success. Since the references may not be readily available, we quote the elegant proof of Feller [6].

"Let it be required to compute $f(\xi)$ of a polynomial of the nth degree $f(x)$, given $f_k=f(x_k)$ for $k=0, 1, \cdots, n$. We note

$$f^1(x) = \begin{vmatrix} f_0 & x_0-\xi \\ f(x) & x-\xi \end{vmatrix} \div (x-x_0) \tag{1}$$

is a polynomial of degree $n-1$, and $f^1(\xi)=f(\xi)$. Hence we are required to find

$f(\xi)$ knowing

$$f_k^1 = \begin{vmatrix} f_0 & x_0 - \xi \\ f_k & x - \xi \end{vmatrix} \div (x_k - x_0), \tag{2}$$

for $k=1, 2, \cdots, n$. Thus the problem is reduced from n to $n-1$. In like fashion the problem is further reduced to $n-2$. We note that $x_k-x_0=(x_k-\xi)-(x_0-\xi)$." We change our notation to that conventional in finite differences and illustrate the process for four points (a, u_a), (b, u_b), (c, u_c), (d, u_d), to determine u_x given $a<x<d$. A table is constructed

$$\begin{array}{llll} u_a & & & a-x \\ u_b & u(a,b) & & b-x \\ u_c & u(a,c) & u(a,b,c) & c-x \\ u_d & u(a,d) & u(a,b,d) \quad u(a,b,c,d) & d-x \end{array}$$

where $a-x$, $b-x$, $c-x$, $d-x$ are called "parts" and

$$u(a,b) = \begin{vmatrix} u_a & a-x \\ u_b & b-x \end{vmatrix} \div (b-a), \qquad u(a,c) = \begin{vmatrix} u_a & a-x \\ u_c & c-x \end{vmatrix} \div (c-a),$$

$$u(a,d) = \begin{vmatrix} u_a & a-x \\ u_d & d-x \end{vmatrix} \div (d-a), \quad u(a,b,c) = \begin{vmatrix} u(a,b) & b-x \\ u(a,c) & c-x \end{vmatrix} \div (c-b),$$

$$u(a,b,d) = \begin{vmatrix} u(a,b) & b-x \\ u(a,d) & d-x \end{vmatrix} \div (d-b),$$

$$u(a,b,c,d) = \begin{vmatrix} u(a,b,c) & c-x \\ u(a,b,d) & d-x \end{vmatrix} \div (d-c) = u_x.$$

The iteration is discontinued as soon as the desired number of significant figures is determined.

We stress the peculiar advantages of the method. The intervals need not be equal; x may be anywhere among the arguments, that is, the interpolation may be central or non-central; it is well adapted to machine calculations; no differencing is required; and finally it may be used for inverse interpolation as well as direct interpolation merely by interchanging the argument and the function. Of course the method is not intended for routine calculations in the construction of tables or for the purposes of subtabulation.

We give only one example in the mathematics of finance, a problem involving the yield of a bond, although the method is applicable wherever interpolation is needed. Thus the power of the usual short tables of interest functions given in the typical text is greatly increased making unnecessary seven place logarithms so frequently advocated. The method is quite effective in the solution

of any equation $g(x)=0$. A $5\frac{1}{2}\%$ bond, coupons payable semi-annually, redeemable in 25 years at 110 is purchased for 105; what is the yield? [7, pp. 190, 226]. Denote the yield for the half year by i, and the equation we must solve is

$$f(i) = 5 + (2.75 - 110i)a_{\overline{50}|i} = 0.$$

The table will be

1.75					−32.34150
2.00	2.5538343				−22.28299
2.50	2.6371542	6|612588			− 5.00000
3.00	2.7243090	6|746793	6|6001		9.15137
3.50	2.8150126	6|889149	6|6618	552	20.80118

Hence $i=2.66552\%$, the extra figures are carried since errors accumulate rapidly in successive multiplications. It should be noted that in iterating the process one multiplies only where the figures disagree. Since Todhunter insisted that the answer was 2.6656%, $f(4\%)=30.44560$ was added to the table resulting in $i=2.665651\%$. Using $f(2.665651\%)=-.001176$ and $f(2.6656\%)=.000314$, we obtain by linear interpolation $i=2.665611\%$ correct to seven significant figures. If an interval of $\frac{1}{4}\%$ is taken using 2%, $2\frac{1}{4}\%$, $2\frac{1}{2}\%$, $2\frac{3}{4}\%$, and 3%, then $i=2.665607\%$ by Aitken's method. Most texts give values only at intervals of $\frac{1}{2}\%$ after $2\frac{1}{2}\%$. For an interval of $\frac{1}{2}\%$ our experience shows that at most an error of 2 or 3 will occur in the fifth significant figure in a bond yield problem. Even the utility of a bond table is extended by Aitken's method.

Other uses are finding $(1+i)^n$, v^n, $a_{\overline{n}|}$, $s_{\overline{n}|}$, if the value of i is not given in the tables. Generally, it is advisable to start with five points. One might think it worthwhile to make a change in origin and scale, but in our experience the extra time spent was not justified. Inverse interpolation will be rather difficult for most students and a little patience is required. It is best to start a class on direct interpolation with equal intervals. In statistics, Aitken's method makes it easy to extend the linear interpolation formula employed in determining the median; essentially one solves the equation $f(x)=.5$, assuming that the probability function is continuous.

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INTRODUCING $e=2.718+$

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Textbooks of calculus usually introduce the number e by considering the limit of $(1+1/n)^n$. Such treatment frequently omits many of the essential steps and leaves the student not much enlightened. An alternative approach is suggested in this note which leads to 2.718 by a more natural and elementary consideration.

The most familiar exponential is $y=2^x$. The slope of this curve at the point (0, 1) can be found by interpolation using a four place table as 0.70, while the slope of the exponential $y=3^x$ comes out 1.10. These two results suggest the query: is there some number between 2 and 3, for which the corresponding slope is 1? Let us assume there is and call this number e, and thus assume that when Δx is the smallest value for which we can interpolate we shall have $y=e^x$ yielding $\Delta y/\Delta x=1$.

For Δy we have $e^{\Delta x}-1$ at the point (0, 1), and so we must solve the equation $(e^{\Delta x}-1)/\Delta x=1$, or $e^{\Delta x}=1+\Delta x$, whence $\Delta x(\log e)=\log(1+\Delta x)$.

From a seven place table we have

$$\log(1-.001) = -.000\ 4345$$

$$\log(1+.001) = +.000\ 4341$$

and these items suggest that when Δx is nearer to zero than either plus or minus .001, we should use the intermediate decimal, .0004343, and the interpolation formula

$$\log(1+\Delta x) = .000\ 4343\,(\Delta x/.001).$$

Putting this into the equation for $\Delta x(\log e)$ above we obtain $\log e=.4343$, whence $e=2.718+$.

If a four place table is used, a less accurate value, $2.7+$, is obtained. But the assumption that there is a number e such that for $y=e^x$, $dy/dx=e^x$ exactly, makes it possible to obtain the familiar expansion for e^x, from which the value of e can be determined as accurately as desired.

TANGENT LINES AND PLANES

F. H. YOUNG and J. L. ERICKSEN, Oregon State College

Although the problem of finding a tangent line to a conic or a tangent plane to a quadric presents no great difficulty, the problem can be solved by a method even simpler than that ordinarily seen.

Consider a given circle and a fixed point on the circle. Write the equation of a circle with its center at the point and of radius less than that of the given circle. If the two equations are normalized, the difference between the equations of the first and second circles is then the radical axis, a straight line passing through the points of intersection of the circles. Now, if the radius of the second circle were decreased, the intersection points would approach the tangent line at the fixed point. This suggests the following method: solve simultaneously the equation of the given circle and the equation of the null circle at the fixed point. The method is so simple that in many cases the answer may be obtained by inspection.

In the case of an ellipse, much the same method may be used. At a point on the given ellipse, consider a null ellipse of the same orientation whose eccentricity is the same as that of the given ellipse. Then the difference between the two equations will represent the tangent line.

In the case of a hyperbola, we can make use of the degenerate hyperbola of the same eccentricity and orientation as the given hyperbola. Geometrically, this means the equation of two straight lines, parallel to the asymptotes, intersecting at the fixed point on the given hyperbola. Again the difference is the required tangent.

For a parabola, we use the equation of a degenerate parabola with vertex at the fixed point and axis parallel to that of the given parabola. The method still works.

Geometrically, our method consists of employing a degenerate form of the conic section of the same eccentricity and orientation as that of the curve on which the fixed point lies. Algebraically, we subtract the equation satisfied by the coördinates of the point from the given equation, thus obtaining a linear equation.

The general equation of the conic, and this includes the degenerate, null, and imaginary cases, is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \tag{1}$$

Consider the point (u, v) satisfying the above equation. Then our null or degenerate form of the same eccentricity will be

$$a(x - u)^2 + b(x - u)(y - v) + c(y - v)^2 = 0. \tag{2}$$

Expanding and subtracting, we obtain

$$(2au + bv + d)x + (bu + 2cv + e)y - (au^2 + buv + cv^2 - f) = 0.$$

It is then easily verified that this is the equation of the line tangent to the given curve at the point (u, v). If equation (1) represents a null circle or null ellipse, it will be identical with equation (2). This is the only case in which a tangent line cannot be obtained. It is interesting to note that the method works equally well for points on imaginary conics.

An especially simple example of this method is the case of the tangent line at the origin to a conic passing through the origin. Consider, then, the conic

$$ax^2 + bxy + cy^2 + dx + ey = 0.$$

The desired tangent is merely $dx+ey=0$.

It is apparent that this method may be extended to the case of the general second degree surface in n dimensions to obtain the equation of the tangent plane.

RATIONAL SOLUTIONS OF A CERTAIN TRIGONOMETRIC EQUATION

L. J. MATTESON, Mount Vernon, N.Y.

To students of trigonometric equations it may be of interest to find the relationship between the constants of the trigonometric equation

$$a \cos \omega + b \sin \omega = c \tag{1}$$

such that the solutions for $\cos \omega$ and $\sin \omega$ will be rational numbers.

Certain values of the constants can be found by inspection. Thus, when one function $\sin \omega$ or $\cos \omega$ is zero the other must be $+1$ or -1. Then $c = \pm a$, or $c = \pm b$, which are the relations existing for quadrantal angles, are the relations which will assure rational values for the two functions. However, more general conditions on the constants of equation (1) can be found by turning to a general solution of the given equation.

In the April 1947 issue of this MONTHLY, R. W. Wagner uses effectively the substitution $t = \tan(\omega/2)$ which leads to the following values for $\sin \omega$, $\cos \omega$ and $\tan(\omega/2)$:

$$\sin \omega = 2t/(1 + t^2), \qquad \cos \omega = (1 - t^2)/(1 + t^2),$$
$$\tan \frac{\omega}{2} = \sin \omega/(1 + \cos \omega). \tag{2}$$

This shows that the functions $\sin \omega$ and $\cos \omega$ are rational if and only if $\tan(\omega/2) = t$ is rational.

By means of (2) equation (1) becomes the quadratic in t:

$$(a + c)t^2 - 2bt + c - a = 0. \tag{3}$$

Let $4k^2$ represent the discriminant of this equation. The values of t, which are the roots, will be rational if and only if $4k^2$ is zero or a perfect square. This leads

to the condition

$$a^2 + b^2 - c^2 = k^2, \qquad (k = 0, \text{ or an integer}). \tag{4}$$

At the same time it is noted that the values of t are

$$t = (b \pm k)/(a + c). \tag{5}$$

From the conditions in (4) it is evident that $a^2+b^2 \geqq c^2$, that the constants a, b and c occur as squares, also that a and b can be interchanged. Let the equation (1) which contains the constants a, b and c be denoted by the symbol (a, b, c). Then (4) which is the necessary and sufficient condition for rational values of $\sin\omega$ and $\cos\omega$ in the equation (a, b, c) will also assure rational values of the functions in the seven related equations which may be designated by the symbols (b, a, c), $(-a, b, c)$, $(-b, a, c)$, $(-a, -b, c)$, $(-b, -a, c)$, $(a, -b, c)$ and $(b, -a, c)$.

The equation (4) can be rewritten so as to introduce the positive integer N;

$$N = a^2 + b^2 = c^2 + k^2. \tag{6}$$

Since the problem now may be reduced to finding the integers N which may be written as the sum of two squares in more than one way, a notation for N equivalent to equation (6) may prove convenient:

$$N = (a, b; c, k). \tag{7}$$

Such numbers N, as implied in Dickson's *The History of the Theory of Numbers*, Vol. II, p. 228, can be found by the following method: all the products and doubles of these products of two or more of the prime numbers of the form $4n+1$ (n an integer) including multiple powers of these numbers are numbers that may be expressed as the sum of two squares in more than one way. Thus 5, 13, 17, 29 $\cdots$ are such prime numbers. From these a few of the products which may be the numbers N are 25, 50, 85, 100, 125, 130, $\cdots$. By the notation in (6) a few numbers for N are $N=25=(3, 4; 5, 0)$, $N=50=(1, 7; 5, 5)$, $N=65=(1, 8; 4, 7)$, $\cdots$.

In $(a, b; c, k)$ there are four basic sets of constants that may be used in (1) to give rational solutions. They are (a, b, c), (a, b, k), (c, k, a) and (c, k, b). Each set has the seven related equations described above, making 32 possible combinations of constants for each N, if a, b, c and k are distinct. As an example, from $N=65=(1, 8; 4, 7)$, the four basic sets are (1, 8, 4), (1, 8, 7), (4, 7, 1) and (4, 7, 8). Of these (1, 8, 4) will represent the equation

$$\cos\omega + 8\sin\omega = 4. \tag{8}$$

Substitution in (5) leads to $t=3$ and $\sin\omega=3/5$, $\cos\omega=-4/5$ or $t=1/5$ and $\sin\omega=5/13$, $\cos\omega=12/13$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 836. *Proposed by W. R. Ransom, Tufts College*

If a right circular cone is cut in an elliptical section by a plane perpendicular to a given element of the cone, then the center of curvature of the vertex of the ellipse lying on the given element lies on the axis of the cone.

E 837. *Proposed by Roy Dubisch, Fresno State College*

Find the sum and interval of convergence of the series

$$\sum_{n=0}^{\infty} (a + nd)x^n.$$

E 838. *Proposed by H. T. R. Aude, Colgate University*

By a translation of axes and a suitable choice for the scale of the ordinates the general polynomial equation of the fifth degree can be taken in the form

$$y = x^5 + ax^3 + bx^2 + cx + d.$$

Assume that there exist four bend points. Find the polynomial cubic equation which is satisfied by these four points. Also find the equation of the parabola with axis parallel to the y-axis which will pass through the three points of inflection.

E 839. *Proposed by E. D. Schell, Office of the Comptroller, United States Air Forces*

Given $S_n = n \cdot 1 + (n-1)2 + \cdots + 2(n-1) + 1 \cdot n$. Show that $S_n = C(n+2, 3)$.

E 840. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Inscribe three equal circles (A), (B), (C) in the corresponding interior angles of a triangle ABC such that we may insert between (B) and (C) a chain of tangent circles equal to (B) and (C) and all touching side BC, and similar chains between (C) and (A) and between (A) and (B). What is the condition for possibility of solution, and how many solutions are there for a given triangle ABC?

SOLUTIONS

A Ruler Construction

E 793 [1947, 545]. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

With straight edge alone construct a hexagon which can possess both an inscribed and a circumscribed conic.

Solution by R. T. Hood, University of Wisconsin. Let us show how, given five lines, a sixth may be determined so that the resulting hexagon shall possess both an inscribed and a circumscribed conic.

Draw any four lines a, b, c, d, forming a quadrilateral with vertices $ab=1$, $bc=2$, $cd=3$, $da=4$. Let the sides a and c intersect in a point P and let b and d intersect in a point Q. Let any fifth line be drawn intersecting c in a point 5 and d in a point 6. Call this line e. The problem is now to determine a sixth line to go with a, b, c, d, e.

Draw the diagonals 13 and 24 of the quadrilateral; call their point of intersection 7. Draw the lines 57 (intersecting a in a point 8) and 67 (intersecting c in a point 9). Draw the line 89 and call it f. Let it intersect the line e in a point R. Then the figure 489256 (or *afbced*) is the desired hexagon.

For, by construction, the three diagonals (24, 58, 69) of the hexagon are concurrent at 7. Hence, by the converse of Brianchon's theorem, there is a conic having the sides of the hexagon as tangents. Now lines 13, 85, and 96 are also concurrent at 7, and these are lines joining corresponding vertices of the two triangles 189 and 356. Hence, by Desargues' theorem, the corresponding sides of these two triangles intersect in three collinear points. That is, a and c intersect in P, b and d intersect in Q, e and f intersect in R, and P, Q, and R are collinear. But a and c, b and d, e and f are simply pairs of opposite sides of the hexagon. Hence, by the converse of Pascal's theorem, there is a conic having as six of its points the six vertices of the hexagon.

Thus this hexagon can have both an inscribed and a circumscribed conic.

Also solved by Frederick Bagemihl and the proposer.

Associated Conjugates

E 800 [1948, 30]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The polar planes, with respect to a tetrahedron, of the isotomic conjugates of a set of collinear points are coaxal.

Note. If the cevians AP, BP, CP, DP, for a tetrahedron $ABCD$, of a point P, cut the planes of the faces BCD, CDA, DAB, ABC, in A', B', C', D', then the isotomic conjugate P' of P is the point of concurrency of the lines joining A, B, C, D to the isotomic conjugates A'', B'', C'', D'' of A', B', C', D' in the faces of the tetrahedron.

Solution by L. M. Kelly, University of Missouri. Let a_1, a_2, a_3, a_4 be the tetrahedral coördinates of one of the collinear points. Then $1/a_1$, $1/a_2$, $1/a_3$, $1/a_4$ are the coördinates of its isotomic conjugate. The polar plane of this point is

$a_1x_1+a_2x_2+a_3x_3+a_4x_4=0$. Thus if (a) and (b) are two fixed points of the line, and (c) any third point of the line, the matrix of the coördinates of these points has rank less than three, and hence the matrix of the coefficients of the corresponding polar planes has rank less than three. But this is the condition that the planes be coaxal.

Also solved by the proposer.

Editorial note. This problem extends to the tetrahedron a problem proposed by this editor in the *National Mathematics Magazine* (vol. XX, no. 2, pp. 95–97).

Let (p_1, p_2, p_3, p_4) be tetrahedral coördinates of any point P and let P' be the point $(k_1/p_1, k_2/p_2, k_3/p_3, k_4/p_4)$, where k_1, k_2, k_3, k_4 are fixed non-zero constants. Then we may call P' the *associated conjugate* of P, for the given tetrahedron and the constants k_1, k_2, k_3, k_4. The polar plane, p', of P', with respect to the tetrahedron, may be called the *associated polar* of P, and P the *associated pole* of p'. If P and P' coincide, P will be called a *self-conjugate* point. The following theorems are easily established.

1. The associated polars of a set of collinear points are coaxal, and dually, the associated poles of a set of coaxal planes are collinear.

2. The associated polars of two points collinear with a self-conjugate point intersect on the polar plane of that self-conjugate point.

3. The associated polars of a set of coplanar points are copunctual, and dually.

4. The common point of a set of copunctual planes is the associated pole of the plane of complanarity of their associated poles, and dually.

5. Cross-ratio is invariant under an *associate correlation.*

6. The tetrahedron determined by four points P, Q, R, S is perspective with the tetrahedron determined by their associated polars p', q', r', s', the center and plane of perspectivity being corresponding associated pole and polar.

7. The locus of points lying on their own associated polars is the conicoid Σ: $x_1^2/k_1+x_2^2/k_2+x_3^2/k_3+x_4^2/k_4=0$.

8. A point and its associated polar are pole and polar with respect to the conicoid Σ.

9. A point and its isotomic conjugate are associated conjugates for the constants 1, 1, 1, 1. A point and its isogonal conjugate are associated conjugates for the constants $A_1^2, A_2^2, A_3^2, A_4^2$, where A_1, A_2, A_3, A_4 are the areas of the faces of the given tetrahedron.

10. The polars of the isotomic conjugates of any two points collinear with the centroid of the tetrahedron are parallel.

Bidding Sequences

E 801 [1948, 95]. *Proposed by J. P. Ballantine, University of Washington*

How many different bidding sequences are possible in one hand of Bridge?

Solution by Roger Lessard, Ecole Polytechnique, Montreal. Consider any specific set of n bids. The bidding will cease if and only if three passes occur in succession. Since a bid can be made in place of the first, second, or third pass, there are three times as many ways of continuing the bidding as of finishing it. Let us count the ways the bidding can be finished. The three passes can be preceded by any one of the following sequences: *B*, *BD*, *BDR*, *BPPD*, *BPPDR*, *BDPPR*, *BPPDPPR*. After each bid there are, then, seven ways of finishing the bidding, and twenty-one ways of continuing the bidding. Thus any specific set of n bids can be made in $7(21^{n-1})$ ways. But, because there are five suits and seven levels, there are 35 possible bids. Thus the number of sets of n bids is $C(35, n)$. Now

$$\sum_{n=1}^{35} C(35, n)(7)(21^{n-1}) = (7/21)\sum C(35, n)(21^n)$$

$$= [(1 + 21)^{35} - 1]/3 = (22^{35} - 1)/3.$$

But each player can make the first bid. Therefore the number of bidding sequences is $4(22^{35}-1)/3$. If we count the case in which no bid is made we have $[4(22^{35})-1]/3$.

Also solved by J. F. Darling, Karl Itkin, and the proposer. Itkin said that, in expanded form, the final result above is

128,745,650,347,030,638,120,231,926,111,609,371,363,122,697,577.

The Sum of Cubes

E 802 [1948, 95]. *Proposed by E. A. Nordhaus, Michigan State College*

(1) Find the smallest positive integer N having the property that the sum of its digits does not divide the sum of the cubes of its digits.

(2) Find the two consecutive positive integers each of which equals the sum of the cubes of its digits.

Solution by J. M. McLynn, Washington, D.C. (1) Since $a+b$ is a factor of a^3+b^3 no two digit number will satisfy the problem. However, $a+b+c$ is not a factor of $a^3+b^3+c^3$, and on second trial (since we need not consider numbers containing zeros) we find 112 to be the required number.

(2) If the required numbers are N and $N+1$, then N must end in 0, because 0 and 1 are the only cubes whose difference is 1. No two digit number will satisfy the problem because $a^3+b^3=10a+b$ (where b must be 0) has no integral solution. The equation $a^3+b^3+c^3=100a+10b+c$ (where a and b are digits and c is 0) has only one solution: $a=3$, $b=7$, $c=0$. Thus the required numbers are 370 and 371. The solution is unique since $a^3+b^3+c^3+d^3=1000a+100b+10c+d$ (where $d=0$ and a, b, c are digits) has no satisfactory solution. There can be no five digit solution, since $5(9^3)$ is only a four digit number.

Also solved by Colin Blyth, W. E. Buker, R. E. Crane, Monte Dernham, L. A. Dwight, Frank Herlihy, Frank Kocher, R. S. Lehman, Roger Lessard,

Leo Moser, C. S. Ogilvy, George Shapiro, C. W. Trigg, G. W. Walker, and the proposer.

Moser pointed out that there are just four numbers having the property of being equal to the sum of the cubes of their digits: 153, 370, 371, 407. This fact is given in Ball-Coxeter, *Mathematical Recreations and Essays*, p. 13, 1944 edition. There a further reference is given to *Sphinx*, 1937, pp. 72, 87.

Triangle with Squares of Sides in Arithmetic Progression

E 803 [1948, 95]. *Proposed by J. H. Butchart, Arizona State College*

If the squares of the sides of a triangle form an arithmetic progression, the line joining the centroid and the symmedian point is parallel to one side of the triangle.

I. *Solution by L. M. Kelly, University of Missouri.* If b is the middle side in the progression the sum of the squares of the sides is $3b^2$. The distance to the side b from the symmedian point is given by

$$2\Delta b/(a^2 + b^2 + c^2) = 2\Delta/3b = h_b/3,$$

where Δ is the area of the triangle and h_b is the length of the altitude on side b. But the distance of the centroid from side b is likewise $h_b/3$.

II. *Solution by the Proposer.* Let A, B, C, K, G be position vectors of the vertices, symmedian point, and centroid of the triangle. Then

$$K = \frac{a^2A + b^2B + c^2C}{a^2 + b^2 + c^2} = \frac{(b^2 - d)A + b^2B + (b^2 + d)C}{3b^2}$$

$$= \frac{A + B + C}{3} + \frac{d}{3b^2}(C - A) = G + \frac{d}{3b^2}(C - A).$$

Thus the join of K and G is parallel to the join of C and A.

Also solved by Yu Yuan Chin, K. W. Crain, L. A. Dwight, Frank Herlihy, B. R. Leeds, D. W. Matlack, N. C. Scholomiti, P. D. Thomas, and C. W. Trigg.

Editorial Note. For a similar problem where the tangents of the angles of the triangle are in arithmetic progression see E 259 [1937, 541]. For the case where the sines of the angles (or the sides) of the triangle are in arithmetic progression see E 411 [1940, 708].

The Euler Line of a Tetrahedron

E 805 [1948, 96]. *Proposed by N. A. Court, University of Oklahoma*

If two coplanar edges of a tetrahedron are each equal to the respectively opposite edge, the remaining two opposite edges are each coplanar with the Euler line of the tetrahedron.

Solution by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C. (Numbers in parentheses refer to articles in the proposer's *Modern Pure Solid*

Geometry, Macmillan, 1935.) The bimedians of a tetrahedron meet in the centroid. Under the conditions given, one of the bimedians is also a bialtitude (201). The circumcenter is the point common to the mediators of the six edges. But two mediators must have in common the bimedian which is also a bialtitude, whence the circumcenter lies on this bialtitude. That is, the bimedian which is also a bialtitude is the Euler line of the tetrahedron.

Also solved by L. M. Kelly, C. W. Trigg, and the proposer.

The proposer established the more complete theorem: *In a tetrahedron* (T) *consider the bimedian relative to a pair of opposite edges of* (T), *the bialtitude relative to the same pair of edges, and the Euler line of* (T). *If any two of these three lines coincide, the third line coincides with them.* If two of the bimedians of a tetrahedron coincide with the respective bialtitudes, then, by this theorem, the Euler line of the tetrahedron would coincide with two distinct lines. The paradox is resolved by the fact that in this case the centroid and the circumcenter coincide, and the Euler line becomes indeterminate (297, 298).

The Two Pedestrians

E 806 [1948, 158]. *Proposed by Leo Moser, University of Manitoba*

Lewis Carroll once proposed the following problem.

"Two travellers spent from 2 o'clock till 9 in walking along a level road, up a hill, and home again; their pace on the level being x miles per hour, up hill y, and down hill $2y$. Find the distance walked."

In the original problem x and y were given integers. Deduce the solution to the original problem without *a priori* knowledge of what these integers are.

Solution by Monte Dernham, San Francisco. If it is r miles along the level road and h miles up hill, then

$$2r/x + h/y + h/2y = 7,$$

whence

$$4yr + 3xh = 14xy,$$

where r and h are the variables, and x and y unknown integers. Clearly, r and h, which need not be integers, are severally indeterminate; also, their sum is indeterminate except in the single case where $4y = 3x$. Thus the unique answer for a pedestrian is given by $x = 4$, $y = 3$, from which $r + h = 14$; and, presuming that Lewis Carroll proposed a problem having a determinate solution, we conclude that each of the travellers walked 28 miles.

Also solved by R. V. Andree, D. M. Brown, W. E. Buker, W. B. Campbell, Richard Courter, R. E. Crane, J. S. Cromelin, Jerome Drexler, R. L. Greene, Sam Kravitz, C. S. Ogilvy, S. T. Parker, E. D. Rainville, F. Underwood, G. W. Walker, and the proposer.

Several solvers pointed out that the assumption that the original problem has a determinate solution is equivalent to insisting that the average rate on the hill be equal to the rate on the level (for otherwise the total distance would be a

function of the ratio of the length of the hill to the length on the level). Hence the harmonic mean between y and $2y$ is x, or $(4/3)y=x$.

The Lewis Carroll problem appears on p. 1025 of *Complete Works of Lewis Carroll*, Modern Library.

Center of Liquid Pressure

E 807 [1948, 158]. *Proposed by R. V. Andree, University of Wisconsin*

An elliptical endgate of a reservoir is to be mounted with the minor axis parallel to the water's surface and in such a manner that it will turn about a horizontal axis in the plane of the gate. Where should this axis be placed so that the gate will not tend to rotate in either direction when the water level is at a given distance above the top of the gate? Generalize to a gate of any shape.

Solution by S. T. Parker, Kansas State College. Let the equation of the ellipse be

$$a^2x^2 + b^2y^2 = a^2b^2,$$

and the height of the liquid c units above the top of the gate. Let the desired axis be denoted by the equation $y=-z$. We have

$$\Delta A = 2x\Delta y,$$

$$\Delta F = 2wx(c + a - y)\Delta y,$$

$$\Delta M = 2wx(c + a - y)(y + z)\Delta y,$$

which gives us

$$\int_{-a}^{a} 2wx(c + a - y)(y + z)dy = 0.$$

Therefore

$$\int_{-a}^{a} (c + a - y)(y + z)(a^2 - y^2)^{1/2}dy = 0.$$

On evaluation, this leads to the result

$$z = a^2/4(c + a).$$

This can be generalized to a gate of any shape as follows. Let the x axis be taken horizontally through the centroid of the area in question. Let the limits of the area be $y=-b$, $y=+a$, and let H $(\geqq a)$ be the height of the liquid above the centroid of the gate. As before, we obtain

$$\int_{-b}^{a} x(H - y)(y + z)dy = 0,$$

or

$$\int_{-b}^{a} x(H-y)ydy = -\int_{-b}^{a} x(H-y)zdy.$$

Now

$$\int_{-b}^{a} xydy = 0.$$

Therefore

$$-\int_{-b}^{a} xy^2dy = -Hz\int_{-b}^{a} xdy,$$

or

$$z = \frac{\int_{-b}^{a} xy^2dy}{H\int_{-b}^{a} xdy}.$$

This is equivalent to saying that $z=k^2/H$, where k is the radius of gyration of the area of the gate with respect to the horizontal centroidal axis.

Also solved by D. M. Brown, W. B. Campbell, Roger Lessard, Leo Moser, C. S. Ogilvy, Arthur Pancoe, and the proposer.

Editorial Note. The problem of locating the center of liquid pressure is treated in many elementary texts. Thus Moser mentioned *Calculus*, L. M. Kells, pp. 337–338, and Ogilvy mentioned *Differential and Integral Calculus*, Ross R. Middlemiss, p. 311, 2nd (1946) edition.

An Interesting Perfect Cube

E 808 [1948, 158]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the number

$$N = 19000458461599776807277716631$$

is a perfect cube and that the twenty-eight numbers which are formed by cyclic permutations of its digits are all divisible by the cube root of N.

Solution by C. W. Trigg, Los Angeles City College. In the conventional manner we find

$$M = N^{1/3} = 2668425111.$$

We also observe that

$$10^{29} - 1 = 9(2668423111)(4163924028879808001).$$

Represent N by $a_1a_2a_3 \cdots a_{29}$, N_1 by $a_2a_3 \cdots a_{29}\,a_1$, N_2 by $a_3 \cdots a_{29}a_1a_2$, *etc.* Then

$$N_1 = 10N - a_1(10^{29}) + a_1 = 10N - a_1(10^{29} - 1),$$

$$N_2 = 10N_1 - a_2(10^{29} - 1),$$

etc. Since N and $10^{29}-1$ are each divisible by M, then N_1 is also. It follows that all the N_i are divisible by M.

In general, if N has p digits and M is a divisor of 10^p-1, then M is a divisor of the cyclic permutations of N. The eight smallest values of N having this property are tabulated below:

p	$10^p - 1$	N	M
2	$3^2 \cdot 11$	27	3
3	$3^3 \cdot 37$	729	9
4	$3^2 \cdot 11 \cdot 101$	1331	11
5	$3^2 \cdot 41 \cdot 271$	68921	41
6	$3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$	250047	63
		456533	77
		753571	91
		970299	99

Also solved by D. M. Brown, J. F. Darling, Roger Lessard, and Alvin and Martin Milgram (jointly).

A Divergent Series

E 809 [1948, 158]. *Proposed by P. L. Chessin, New York, N. Y.*

Show that the following series diverges:

$$\sum_{n=1}^{\infty} n^{-(n+1)/n}$$

I. *Solution by P. A. Clement, University of California at Los Angeles.* For a series, $\sum a_n$, of positive monotone decreasing terms, the well known theorem of Olivier (see Knopp, *Theory and Application of Infinite Series*, Blackie, 1928, p. 124) provides as a necessary condition for convergence that $\lim_{n\to\infty} na_n = 0$. By an easy calculation we find this limit to be 1 for the given series of positive monotone decreasing terms; hence the series diverges.

As a second solution, it is readily established that half the harmonic series may be used in a comparison test; *i.e.*, the given series is termwise greater than $\sum 1/2n$, and thus must diverge.

II. *Solution by D. W. Matlack, North American Aviation, Inc.* We use the well known theorem (see Hardy, *Pure Mathemtaics*, 6th edition, p. 309): If $\sum u_n$ is divergent and u_n/v_n tends to a limit other than zero as $n\to\infty$ then $\sum v_n$ is divergent. In the present case let $u_n=n^{-1}$, $v_n=n^{-(n+1)/n}$. Then $u_n/v_n=n^{1/n}\to 1$.

Also solved by J. O. Blumberg, W. G. Brady, W. H. Breen, D. M. Brown, R. L. Clayton and Eugene McLachlan (jointly), Ragnar Dybvik, Karl Itkin, M. S. Klamkin, Roger Lessard, Michael Martino, Jr., Norman Miller, Leo Moser, C. S. Ogilvy, W. Seidel, George Shapiro, E. J. Stulken, F. Underwood, W. R. Van Voorhis, and the proposer.

Editorial Note. As Miller pointed out, this problem occurs in Knopp, *Theory and Application of Infinite Series*, Blackie, 1928, p. 119, ex. h, where another proof of divergence is indicated. The problem is also found as ex. 34, p. 125 of the same work, where the application of Olivier's theorem is indicated. Lessard established the more general theorem: The series $\sum 1/n^{f(n)}$ is divergent if $\lim_{n\to\infty} f(n)=1$.

Series of Involutes

E 810 [1948, 159]. *Proposed by J. H. Butchart, Arizona State College*

Find the sum of the infinite series consisting of the following terms: the radius of the unit circle, the arc intercepted by the central angle x, the arc of the involute corresponding to this arc, the arc of the involute constructed on the first involute, and so on indefinitely.

Solution by Leo Moser, University of Manitoba. Let $S_n(x)$ be the length of the nth involute. It is easily shown that $S_1(x)=\int_0^x t\,dt$ and $S_n(x)=\int_0^x S_{n-1}(t)\,dt$. Hence $S_n(x)=x^{n-1}/(n+1)!$ and the required sum is e^x.

Also solved by W. G. Brady, D. M. Brown, Karl Itkin, D. W. Matlack, C. S. Ogilvy, R. A. Rosenbaum, F. Underwood, and the proposer.

Editorial Note. The relation $dS_n/dt=S_{n-1}$ is easily established by differential methods and may be found, *e.g.*, in Williamson, *Differential Calculus*, 1927, p. 304. Rosenbaum pointed out that $S_n(x)=x^{n+1}/(n+1)!$ appears in the solution of E 635 [1945, 161].

A given arc AB has two particular involutes, depending on whether end A or end B of an unwinding thread AB generates the curve. If we regard arc AB as a directed arc then the former case will lead to what we shall call the *direct involute* of arc AB, and the second case to what we shall call the *indirect involute* of arc AB. We now direct an involute by the direction in which it is traced by the moving end of the unwinding thread. If, in the given problem, we use the sequence of direct involutes we obtain the sum e^x. On the other hand, if we use the sequence of indirect involutes we obtain the sum $\tan x+\sec x$. Ogilvy found this second sum, which is immediately apparent from a figure.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4315. *Proposed by Albert Wilansky, Brown University*

Consider the Clairaut differential equation

$$y = px + f(p),$$

where $p=y'$ and f has a derivative. Prove that if f' is monotone the singular solution has exactly one point in common with any particular solution.

4316. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let there be given a skew quadrilateral having the sum of one pair of opposite sides equal to the sum of the other pair. Then, (1) there are infinitely many spheres tangent to all four sides of the quadrilateral, the locus of the centers being a straight line Δ, (2) the points of contact of any one of the spheres lie on a plane perpendicular to Δ, (3) the sides of the quadrilateral belong to a hyperboloid of revolution which envelopes all the spheres and has Δ as axis.

4317. *Proposed by Leo Moser, University of Manitoba*

Let G be an Abelian group and A a subset of order n, such that $a\in A$ implies $a^{-1}\in G-A$. Consider the n^2 (not necessarily distinct) elements of G of the form a_ia_j, a_i and a_j elements of A. Show that of these n^2 elements at most $\binom{n}{2}$ are elements of A.

4318. *Proposed by H. F. Sandham, Trinity College, Ireland*

Show that

$$\frac{1}{(\sinh \pi)^2}+\frac{1}{(2\sinh 2\pi)^2}+\frac{1}{(3\sinh 3\pi)^2}+\cdots = \frac{2}{3}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\cdots\right)-\frac{11\pi^2}{180}.$$

4319. *Proposed by Paul Erdös, Syracuse University*

If p is a prime greater than 3 and if $n=(2^{2p}-1)/3$, show that 2^n-2 is divisible by n.

SOLUTIONS

A Locus Problem

4170 [1945, 400]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The powers of the vertices A, B, C of a given triangle with respect to the circles (ω_1), (ω_2), (ω_3) are respectively (ka^2, kb^2, kc^2), (kb^2, kc^2, ka^2), (kc^2, ka^2, kb^2), where a, b, c are the lengths of the sides of ABC. Find the loci of the centers ω_i as k varies. (2) The circumcenter of ABC is the centroid of triangle $\omega_1\omega_2\omega_3$ which remains similar to itself. (3) The straight line $\omega_2\omega_3$ is perpendicular to the join of the centroid and Lemoine point of ABC.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** We choose triangle ABC as the triangle of reference in a system of normal coördinates. We let

$$C \equiv ayz + bzx + cxy, \qquad \Delta \equiv ax + by + cz.$$

Then $C=0$ and $\Delta=0$ are the equations of the circumcircle of ABC and of the line at infinity respectively. The equations of circles (ω_1), (ω_2), (ω_3) are then

$$C - k\Delta(a^3x + b^3y + c^3z)/abc = 0, \qquad (\omega_1)$$

$$C - k\Delta(ab^2x + bc^2y + ca^2z)/abc = 0, \qquad (\omega_2)$$

$$C - k\Delta(ac^2x + ba^2y + cb^2z)/abc = 0, \qquad (\omega_3)$$

The coördinates of the centers ω_1, ω_2, ω_3 are

$$\begin{aligned}
x_1 &= R\cos A + Rk(-a^3 + b^3\cos C + c^3\cos B)/abc,\\
y_1 &= R\cos B + Rk(a^3\cos C - b^3 + c^3\cos A)/abc,\\
z_1 &= R\cos C + Rk(a^3\cos B + b^3\cos A - c^3)/abc;\\
x_2 &= R\cos A + Rk(-ab^2 + bc^2\cos C + ca^2\cos B)/abc,\\
y_2 &= R\cos B + Rk(ab^2\cos C - bc^2 + ca^2\cos A)/abc,\\
z_2 &= R\cos C + Rk(ab^2\cos B + bc^2\cos A - ca^2)/abc;\\
x_3 &= R\cos A + Rk(-ac^2 + ba^2\cos C + cb^2\cos B)/abc,\\
y_3 &= R\cos B + Rk(ac^2\cos C - ba^2 + cb^2\cos A)/abc,\\
z_3 &= R\cos C + Rk(ac^2\cos B + ba^2\cos A - cb^2)/abc;
\end{aligned}$$

where R equals the circumradius of ABC. For varying k, these represent parametric equations of circumdiameters. The radical axes of the circumcircle and the circles (ω_1), (ω_2), (ω_3) have equations

$$\begin{aligned}
a^3x + b^3y + c^3z &= 0,\\
ab^2x + bc^2y + ca^2z &= 0,\\
ac^2x + ba^2y + cb^2z &= 0,
\end{aligned}$$

respectively. Hence ω_1, ω_2, ω_3 describe diameters perpendicular to their respec-

* Translated and checked by O. J. Ramler, Catholic University of America.

tive radical axes. As k varies from 0 to ∞, the triangle $\omega_1\omega_2\omega_3$ varies, beginning as a point, $(R\cos A, R\cos B, R\cos C)$, the circumcenter, and increasing in magnitude, its sides remaining parallel to themselves. Thus the circumcenter O is the center of homotheticity. Moreover, we have

$$x_1 + x_2 + x_3 = 3R\cos A, \qquad y_1 + y_2 + y_3 = 3R\cos B,$$
$$z_1 + z_2 + z_3 = 3R\cos C.$$

Hence O is also the center of gravity of the triangle $\omega_1\omega_2\omega_3$ for every value of k. Since the coördinates of the centers $\omega_1, \omega_2, \omega_3$ given above are actual coördinates (and not relative) the point at infinity on the line $\omega_2\omega_3$ has coördinates,

$$x_2 - x_3 = -a(b^2 - c^2) + b(c^2 - a^2)\cos C + c(a^2 - b^2)\cos B,$$
$$y_2 - y_3 = a(b^2 - c^2)\cos C - b(c^2 - a^2) + c(a^2 - b^2)\cos A, \tag{4}$$
$$z_2 - z_3 = a(b^2 - c^2)\cos B + b(c^2 - a^2)\cos A - c(a^2 - b^2).$$

From (2) and (3) we find the equation of the radical axis for circles (ω_2) and (ω_3) to be

$$a(b^2 - c^2)x + b(c^2 - a^2)y + c(a^2 - b^2)z = 0. \tag{5}$$

Hence the point given by equations (4) is the point at infinity of a line perpendicular to (5), which joins the centroid to the Lemoine point of ABC. Therefore $\omega_2\omega_3$ is perpendicular to this line.

A Difference Sequence

4247 [1947, 232]. *Proposed by Leon Recht and Martin Rosenman, New York City*

The sequence

$$\{a\} = 1; 2; 1, 2; 2, 1, 2; 1, 2, 2, 1, 2; \cdots$$

is formed by writing down in succession sets of terms starting with 1 and such that every subsequent set is obtained from the preceding set by the substitution $1\to 2; 2\to 1, 2$. Show that: (1) the $(n+1)$th* term a_n is

$$[k(n+1)] - [kn],$$

where $k = \frac{1}{2}(\sqrt{5}+1)$ and $[x]$ is the greatest integer not exceeding x: (2) Each set of terms in $\{a\}$, beginning with the third, is a repetition of the terms of the two preceding sets; (3) $\{a\}$ consists of sets of 2's separated by 1's, such that there are a_n 2's in the $(n+1)$th set of 2's.

Solution by E. P. Starke, Rutgers University. Let $c_1=1$, $c_2=2$, $c_3=3$, $c_4=5$, $\cdots$, $c_{n+1}=c_{n-1}+c_n$, be the Fibonacci sequence. Recalling familiar results we know that c_n/c_{n-1} is the nth convergent to the continued fraction development

* Corrected from the original statement where it appeared as the nth.

of k. We have also

$$(1) \qquad 0 < \frac{c_{2n+2}}{c_{2n+1}} - k < k - \frac{c_{2n+1}}{c_{2n}} < \frac{c_{2n+2}}{c_{2n+1}} - \frac{c_{2n+1}}{c_{2n}} = \frac{1}{c_{2n}c_{2n+1}},$$

$$(1') \qquad 0 < k - \frac{c_{2n+1}}{c_{2n}} < \frac{c_{2n}}{c_{2n-1}} - k < \frac{c_{2n}}{c_{2n-1}} - \frac{c_{2n+1}}{c_{2n}} = \frac{1}{c_{2n}c_{2n-1}},$$

$$[c_{2n}k] = c_{2n+1}, \qquad [c_{2n-1}k] = c_{2n} - 1.$$

Let u, v be any integers. Then if $u/v > k$, $c_{2n}u - c_{2n+1}v = 0$ if and only if $u = tc_{2n+1}$, $v = tc_{2n}$; $c_{2n}u - c_{2n+1}v = 1$ if and only if $u = c_{2n+2} \pm tc_{2n+1}$, $v = c_{2n+1} \pm tc_{2n}$; where t is an arbitrary integer and $n = 1, 2, 3, \cdots$. Thus for any $u > vk$ and for $c_{2n-1} < v < c_{2n+1}$, we have

$$c_{2n}u - c_{2n+1}v \geqq 2;$$

and therefore

$$u/v - c_{2n+1}/c_{2n} \geqq 2/c_{2n}v.$$

From (1)

$$k - c_{2n+1}/c_{2n} < 1/c_{2n+1}c_{2n} < 1/c_{2n}v,$$

and from these last two inequalities

$$u/v - k > 1/c_{2n}v.$$

If for u we take $[vk]+1$, we have

$$(2) \qquad vk - [vk] < 1 - 1/c_{2n}.$$

Now from (1') we have

$$c_{2n} - 1/c_{2n} < c_{2n-1}k < c_{2n}.$$

Obviously,

$$[(v - c_{2n-1})k] < (v - c_{2n-1})k < [(v - c_{2n-1})k] + 1,$$

and by adding,

$$[(v - c_{2n-1})k] + c_{2n} - 1/c_{2n} < vk < [(v - c_{2n-1})k] + c_{2n} + 1.$$

From this we conclude

$$[vk] = c_{2n} + [(v - c_{2n-1})k],$$

because a value of $[vk]$ smaller by 1 would contradict (2). Replacing c_{2n} by $[(c_{2n-1})k]+1$ from (1') we have

$$(3) \qquad [vk] = [c_{2n-1}k] + [(v - c_{2n-1})k] + 1.$$

Let $a_j=[(j+1)k]-[jk]$. By setting $v=c_{2n-1}+1, c_{2n-1}+2, \cdots, c_{2n+1}-1$ in (3) and subtracting the resulting equations, each from the following, we find

$$a_{c_{2n-1}} = [k] + 1 = 2$$

and

$$a_{w'} = a_w, \qquad w' = w + c_{2n-1}, \quad w = 1, 2, \cdots, c_{2n} - 2. \tag{4}$$

On the other hand, if $k>u/v$, a clearly analogous argument leads from (1) and (1′) through

$$vk - [vk] > 1/c_{2n-1}, \qquad c_{2n-2} < v < c_{2n}. \tag{2′}$$

$$[vk] = [c_{2n-1}k] + [(v - c_{2n-1})k], \tag{3′}$$

and finally to

$$a_{c_{2n-1}} = [k] = 1$$

and

$$a_{w'} = a_w, \qquad w' = w + c_{2n-2}, \qquad w = 1, 2, \cdots, c_{2n-1} - 2. \tag{4′}$$

We can combine (4) and (4′) as

$$a_{w'} = a_w, \qquad w' = w + c_n, \qquad w = 1, 2, \cdots, c_{n+1} - 2. \tag{5}$$

If therefore we separate the sequence $\{a\}$ into sets corresponding to the subscripts

$$\begin{array}{llll} 0; & 1; & 2, 3; & \cdots; \\ c_{2n-2} - 1, & c_{2n-2}, & \cdots, & c_{2n-1} - 2; \\ c_{2n-1} - 1, & c_{2n-1}, & \cdots, & c_{2n} - 2; \\ c_{2n} - 1, & c_{2n}, & \cdots, & c_{2n+1} - 2; \\ c_{2n+1} - 1, & c_{2n+1}, & \cdots, & c_{2n+2} - 2; \cdots, \end{array} \tag{6}$$

each set is seen by (5) to be exactly the terms of the two preceding sets taken in order. This shows that the sequences defined in (1) and (2) of the proposal are identical.

Note that the substitution

$$1 \to 2, \qquad 2 \to 1, 2 \tag{S}$$

changes the first set $a_0=1$ into the second, $a_1=2$, and the second, $a_1=2$, into the third, $a_2=1$ and $a_3=2$. Suppose (S) changes each set of (6) up to the jth into the succeeding set. Since it then changes the $(j-2)$th and the $(j-1)$th into the $(j-1)$th and the jth, it must also, because of the result of the previous paragraph, change the jth into the $(j+1)$th. Hence the sequence developed from $a_0=1$, $a_1=2$ by the use of (S) is the sequence $\{a_j\}$ of our previous discussion.

Let the integers s_j be defined by

$$s_j = j + [jk], \qquad j = 1, 2, 3, \cdots,$$

and note that

$$s_{j+1} = j + 1 + [(j+1)k] = s_j + 1 + a_j.$$

We prove that $a_n = 1$ if and only if n is one of the s_j.* This is readily seen to be equivalent to part (3) of the proposal because between the s_jth and the s_{j+1}th values of a (which are equal to 1) there are $s_{j+1} - s_j - 1 = a_j$ values which equal 2.

The proof follows by a slightly modified induction. Let c' represent c_{2n} and suppose

$$s_{c'+h} = s_{c'} + s_h \tag{7}$$

for a particular value of h. Then by (5), provided $h < c_{2n+1} - 1$, we have

$$\begin{aligned} s_{c'+h+1} &= s_{c'+h} + 1 + a_{c'+h} = s_{c'} + s_h + 1 + a_{c'+h} \\ &= s_{c'} + s_h + 1 + a_h = s_{c'} + s_{h+1}, \end{aligned}$$

and also

$$s_{c'+1} = s_{c'} + 1 + a_{c'} = s_{c'} + 2 = s_{c'} + s_1.$$

Hence (7) is true for $h = 1, 2, \cdots, c_{2n+1} - 2$. Now, since

$$s_{c'} = c_{2n} + [c_{2n}k] = c_{2n+2},$$

we have

$$s_{c'+h} = c_{2n+2} + s_h.$$

Similarly we may show

$$\begin{aligned} s_{c_{2n+1}+h} &= s_{c_{2n+1}} + s_h + 1, \\ &= c_{2n+2} - 1 + s_h + 1, \quad h = 1, 2, \cdots, c_{2n+2} - 2, \end{aligned} \tag{7'}$$

so that we may put

$$s' = s_{c_m+h} = c_{m+2} + s_h, \qquad h = 1, 2, \cdots, c_{m+1} - 2,$$

and by (5) we have

$$a_{s'} = a_{s_h}. \tag{8}$$

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* This theorem may be generalized as follows:

$$[(c_{2n} + v)k] - [c_{2n}k] = [vk] + 1$$

if and only if $v = c_{2n-1}j + c_{2n}[jk]$;

$$[(c_{2n+1} + v)k] - [c_{2n+1}k] = [vk]$$

if and only if $v = c_{2n}j + c_{2n+1}[jk]$, $j = 1, 2, \cdots$.

Thus, since every integer greater than 2 can be written as c_m+h with $h \leqq c_{m-1} \leqq c_{m+1}-2$, $m>2$, and since (by direct inspection) $s_1=2$ and $s_2=5$ give $a_s=1$, we conclude that $a_s=1$, for every s, as required.

We noted above that the number of a's between two successive values of a_s is always 1 or 2, but two consecutive a's cannot both be equal to 1 because $[(j+2)k]-[jk] \geqq [2k]=3$. Hence every a_s equals 1 and every other a equals 2 as required.

Editorial Note. C. D. Olds remarks that the above is a particular case of a problem originally proposed by Bernoulli. It was studied by A. Markoff [*Math. Ann.* 19, 1882, pp. 27–36] and more recently by J. V. Uspensky [*Revista Union Mat. Argentina* 11, 1946, pp. 141–154, 165–183, 239–255 and 12, 1946, pp. 10–19]. These latter papers are listed in *Mathematical Reviews*, 8, 1947, pp. 5, 6, 443.

Convergence of Series

4250 [1947, 286]. *Proposed by Richard Bellman, Princeton University.*

If

$$s_n = \sum_{k=1}^{n} a_k, \qquad \sigma_n = \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) a_k,$$

and

$$\sum_{n=1}^{\infty} |\, s_n - \sigma_n \,|^{\alpha} < \infty,$$

for some $\alpha>0$; prove that $\sum_{n=1}^{\infty} a_n$ is convergent.

Solution by N. J. Fine, Washington, D. C. Since $s_n-\sigma_n \to 0$, it is sufficient to prove that σ_n converges. Now

$$\sigma_n - \sigma_{n-1} = \frac{1}{n(n+1)} \sum_{k=1}^{n} k a_k = \frac{s_n - \sigma_n}{n}.$$

Summing for $n=1, 2, \cdots, N$,

$$\sigma_N = \sum_{n=1}^{N} \frac{s_n - \sigma_n}{n}.$$

If $\alpha \leqq 1$, convergence is immediate. If $\alpha>1$, Hölder's inequality yields

$$\sum_{n=1}^{N} \frac{|\, s_n - \sigma_n \,|}{n} \leqq \left\{ \sum_{n=1}^{N} |\, s_n - \sigma_n \,|^{\alpha} \right\}^{1/\alpha} \left\{ \sum_{n=1}^{N} n^{-\beta} \right\}^{1/\beta},$$

where $1/\alpha+1/\beta=1$. Since both factors converge, so does σ_N.

Also solved by R. C. Buck, W. P. DeWitt, G. G. Lorentz, Otto Szasz, J. G. Wendel, and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 W. 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Fundamentals of Statistics. By T. L. Kelley. Cambridge, Harvard University Press, 1947. 16+755 pages. $10.00.

Whoever has heard Professor Truman Lee Kelley knows how eloquently he speaks for those who enter the field of statistics not primarily from an interest in mathematics or in mathematical statistics but from some contact with an application of statistics to education, psychology, or the social sciences. The text is intended for such students; it makes interesting reading for anyone. The author with a distinguished career in statistics, psychology, and education has previously written a book on the subject "Statistical Method (1923)" and a set of well-known tables "The Kelley Statistical Tables" of which a revision promised for 1947 has presumably appeared. In the preface Professor Kelley states: "An alternative title for the book which would emphasize the point of view is Statistics, Its Philosophy and Method. The endeavor has been to place a great emphasis upon the logic and principles underlying the statistical study of phenomena." On the whole the author has been quite successful in his purpose. He has interwoven and interrelated the various topics and ideas into a closely knit fabric producing an over-all pattern of sound judgment, integrating the practical aspects with the many theoretical considerations involved. The text shows real scholarship and a very wide knowledge of both past and recent results. This has been done moreover with a minimum amount of mathematics.

The first ten chapters, the material for an elementary course, 398 pages, require little mathematics beyond college algebra, but in chapter seven the differential occurs, in chapter eight the slope, derivative, integral, and some trigonometry. The remaining four chapters, 225 pages, are advanced in nature. The highlights of the text besides those already indicated are: a discussion of the purposes and uses of statistics, the treatment of significant figures, the care in determining the size of the class interval and class limits in the frequency distribution, and the introduction of measures of variability before measures of central tendency, a bold and welcome departure. Along with this material the author introduces the standard error of the mean and clearly explains the concept of the number of degrees of freedom. This particular chapter, the sixth, illustrates the expert weaving of many strands into a related whole. The question of efficiency and consistency of a statistic is treated from the point of view of the critical ratio, the ratio of a population parameter to the standard error of a statistic estimating the parameter. This is done consistently and emphasis is placed on indicating the reliability of a statistic by computation of its standard error. Again in chapter seven, measures of central tendency are discussed not as isolated entities but in relation to other characteristics of the probability distri-

bution such as kurtosis and skewness. The discussion of the median and the mean is stimulating but, since the concepts of consistency and efficiency are not defined, it is based solely on intuitive reasoning. The chapter on the properties of the normal curve includes material seldom found elsewhere; another treats correlation from the point of view of regression, the bivariate probability function, and the analysis of variance. Here a gem deserves quotation "Galton's humility, after years of collection of data and subtle analysis of the same, in the face of the neat but not involved mathematical derivation, is worthy of note by the social scientists of this day who scoff at mathematical analysis." This enthusiasm for statistics is well communicated. In the same chapter the student is warned of the dangers in the indiscriminate use of biserial r, ϕ, and the tetrachoric coefficient of correlation. Chapter eleven furnishes an authoritative treatment of the reliability of a test, the tetrad difference, and various other topics in psychological testing. Multiple correlation is thoroughly discussed in chapter twelve, a modified Doolittle solution is given for regression planes as well as a solution of the general correlation problem by matrices. Chapter thirteen includes a study of periodicity, lead and lag, the approximate variance of a statistic for N large, moments of quotients, the Pearson system of curves, sampling from a bivariate normal function, and a substantial section, concise and clear on sequential analysis. Chapter fourteen introduces the student to mathematical topics beyond analytic geometry, such as matrices, determinants, moments of some discrete probability functions, the Gamma function, finite differences, and the Euler-Maclaurin sum formula. In chapter fifteen are selected references to statistical tables, tables of coefficients for Lagrangian interpolation, the normal curve, square and cube roots, and an outline of tables which occur in the text. Appendix A has the unusual feature of a mathematical background test in mathematics for elementary statistics with scoring keys, percentile norms, and reliability. The symbols used, a key to all formulas in the text, and a long list of references to original papers and texts constitute Appendix B. Appendix C is a correlation chart. Problems are given only in the first five chapters.

While a photo-offset process has been used, the results are quite clear. Misprints are very few. Formula 6:37 needs a parenthesis, similarly for 6:55, 6:58 should have $N-1$ in the denominator in place of $N+1$, and 7:08 should have $P_{.75}-P_{.25}$ in the denominator. It would be helpful to indicate that the formulas for the percentiles are merely linear interpolation formulas. In chapter five a generalized measure of variability is defined as

$$f = \frac{\Sigma \mid x_i - x_j \mid^m}{N(N-1)}, \qquad i, j = 1 \text{ to } N,\ i \neq j.$$

Then "This function, f, is worthy of investigation for different values of m, $\cdots$ but the writer has found the function rather intractable except for $m=2$." This function for $m=1$ is Gini's mean difference without repetition, and if divided by N^2 instead of $N(N-1)$ is Gini's mean difference with repetition. For $m=4$ the

expected value of $(x_i-x_j)^4$ would be $2(\mu_4-3\mu_2)$, where both μ_4 and μ_2 indicate population values. Thus for $m>2$, it cannot be claimed that f is a measure of variability.

A formula is stated for the standard error of the average deviation from the mean. It is rather useless, since to evaluate it would be difficult for any of the well known distributions. In the comparison of critical ratios for certain measures of skewness, all such measures should be consistent and estimate the same parameter. For example if α_3 in the population does not exist, it still may happen that the percentile measure of skewness may exist. While intuitional techniques are suggestive, they are not substitutes for rigorous mathematical demonstrations. The impression is created that if a curve has a high peak at the mode, then $\alpha_4>3$, certainly not true in general. One confused remark is the following: "Thus the sample mode, defining with maximum likelihood the parent mode, is a maximum likelihood statistic." The sample mode is usually very unreliable, seldom the solution of a maximum likelihood equation.

In chapter nine Paulson's formula is given for the normalization of the F distribution, in slightly different algebraic form, without a reference to its source, Annals of Mathematical Statistics, 1942. A convenient table is furnished by the author for its calculation. The results of Hotelling and Pabst would have strengthened the section on the coefficient of rank correlation. In the treatment of sequential analysis it should be clearer that the theory was developed almost solely by A. Wald. The books by A. Wald, P. Hoel, H. Cramer, and C. E. Weatherburn are not mentioned in the references, since they were published too recently for inclusion. To these should be added the text of S. S. Wilks.

The reviewer feels strongly that the Neyman-Pearson theory of testing hypotheses merited a very prominent position, a regrettable omission. Other omissions are the theory of confidence intervals and methods of sampling. More emphasis should have been placed on exact sampling distributions.

While a few deficiencies have been noted, this book fulfills a real need in the social sciences, psychology, and education. Both teachers and students will find it worthwhile as a text and for reference.

L. A. AROIAN

Analytic Geometry. By D. S. Nathan and Olaf Helmer. New York, Prentice-Hall, Inc. 1947. 402 pages. $3.50.

This text contains 17 chapters with the following headings, in order: Coordinates on a Line; Rectangular Coordinates in the Plane; Angles; Loci; Lines; Circles; Coordinate Transformations; Conic Sections; Conic Sections in Arbitrary Position; Higher Plane Curves; Families of Curves; Parametric Representation; Polar Coordinates; Fitting of Empirical Data; Rectangular Coordinates in Space; Loci in Space; Planes and Lines. Thus 14 chapters are devoted to Plane Analytics and three to Solid Analytics.

Most of the main special features of this book are briefly indicated in the following seven paragraphs, in some of which our comments are added:

1. The discussion of topics is on the whole clear and for the most part fuller than is customary. In the first and third chapters, the discussions of directed line segments and angles seem unnecessarily involved.

2. The problems are plentiful in each chapter and are of great variety. They involve considerable newness, and the number of relatively difficult problems is larger than for most current textbooks on Analytics.

3. The last section of each chapter includes a summary and a list of review problems. These features seem worthwhile.

4. This book contains more figures than most other current texts on the subject—246 in all. These are on the whole well drawn and satisfactory. However, we would have preferred consistent labeling of the origin of coördinates.

5. Novelty of language is not infrequent in this text. For instances, "major axis" and "minor axis" of an ellipse are replaced, respectively, by "transverse axis" and "conjugate axis," as for a hyperbola; the term "discussion" of an equation is replaced by "analysis" of it; a certain rectangle which is customarily associated with a hyperbola is called its "guide rectangle"; and abbreviations that a reader should understand from context are introduced in several places without explanation.

6. The following elements of novelty of presentation seem noteworthy. The authors do not give a two-point form of equation of a straight line. In the chapter on Coordinate Transformations, the authors show how to remove the cross-product term from the general equation of the second degree (in x and y) before discussing the parabola, ellipse, or hyperbola. In the initial treatment of these conics, each conic is considered in just one (standard) position. The subject of polar coordinates is discussed for the first time in Chapter 13; Sections 86 and 88 of this chapter, which deal with rotations about the pole and intersections of curves, are exceptionally good. In the treatment of Solid Analytics, cylindrical and spherical coordinates are not used; nor are transformations of rectangular coordinates in space considered.

7. Formulas and Tables for Reference (pages 369–374) and Answers to Odd-Numbered Problems (pages 375–393) precede the Index (pages 395–402).

Now relative to corrections, aside from several nearly harmless misprints, we noted the following places where the authors' perspective seemed to need modification:

Page 15. The authors say "the nearer two points in the xy plane are to each other, the less will their abscissas, and likewise their ordinates, differ, and conversely." If the two points have the same abscissas (or ordinates), this quotation is clearly not quite appropriate.

Page 137. In discussing sections of a right circular cone by planes, the authors say "A circle appears as a mere special case of an ellipse (namely when $|VA'| = |BV'|$)." In their discussion, this equality is not sufficient to insure that the ellipse be a circle.

Page 249. In Example 1 here, the authors employ a conic with "arbitrary" eccentricity and then find the conic's equation in a form which can not represent

a circle with a positive radius. Thus the word "arbitrary" was mis-used; the case $e=0$, where e stands for eccentricity, should have been excluded.

Page 302. When k is negative (and $a=k/9$, $b=-4k/9$, $c=8k/9$), the three displayed equations in line 20 of this page are all incorrect; each one of them would be corrected by merely changing the sign of its right member. The authors' error seems to be due to the (false) assumption that $\sqrt{k^2}=k$ when k is negative.

We have no doubt but that this book can be used as a textbook in Analytics with satisfaction. However, in order to do this, we would feel it desirable to make lesson plans in advance which would surely include polar coordinates—perhaps at an early place in the course, so as to be able to use to advantage in many places the authors' Theorem 47 on polar rotations—and enough but not too many "hard" problems; and, for some courses, we would desire to add notes for one or two lectures on parts of Solid Analytics which this text omits.

H. A. SIMMONS

Analytic Geometry. By R. S. Underwood and F. W. Sparks. Boston, Houghton-Mifflin Co., 1947. 10+225 pages. $2.75.

This is a well written little book, clear in its explanations, and looks as if the student could be expected to read and absorb much information from it by himself. Too many text-books are so difficult that they degenerate merely to a source of homework problems. This book combines an excellent fund of exercises with clear exposition.

However, everything has its faults in this far from perfect world and this book is no exception.

First, there is a rather serious oversight in the development of equations of curves. The authors starting with the usual definitions of conic sections, for example, and simplifying (involving the squaring of both sides of an equation), arrive at the standard equations—thus proving that any point of the curve satisfies the equation, but they fail to retrace their steps and prove that any point satisfying the equation must be a point on the curve as defined. The two-edged meaning of "equation of a curve" is not brought out. However, a rather inadequate footnote on pp. 29–30, in connection with a much simpler locus problem, mentions that steps should be retraced but that an obvious reversal of steps is usually involved and hence is often omitted.

Secondly, the conic sections and their properties that are so elegantly studied by analytic methods, are neglected. The standard equations are exhibited, but the tangents to these curves are not considered at all, being brushed off with the statement that calculus is needed for their study. This is not a good excuse, since their slopes can be easily derived (by making the points of intersection of the secant coincide), or the formulas for slopes given without proof. It seems somewhat illogical to neglect conic sections and their tangents and yet spend considerable space on curve tracing and asymptotic behavior which do require calculus for full understanding. Moreover, in the discussion of the asymptotes of

a hyperbola, although the authors show the distance between the curve and its asymptote approaches zero, they cannot show (nor in fact do they mention) that the slope of the curve approaches that of the asymptote.

CHARLES WEXLER

NEW BOOKS RECEIVED

Rings and Ideals (The Eighth Carus Mathematical Monograph). By N. H. McCoy. LaSalle, Illinois, The Open Court Publishing Company, 1948. 12+216 pages. $3.00. (Members of the Mathematical Association of America may purchase one copy at $1.75, order to be placed with the Secretary-Treasurer.)

An Introduction to Pure Solid Geometry. Fourth Edition. By G. S. Mahajani. Poona, 1948. 12+106 pages. Rs3-8-0.

Handbook of Elementary Technical Mathematics. Revised Edition. By J. W. Greenwood and M. I. Chriswell, New York, Prentice-Hall, Inc., 1948. 6+186 pages. $2.10.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Kappa Mu Epsilon, Pittsburg Kansas State Teachers College

Kansas Alpha Chapter of *Kappa Mu Epsilon* holds two initiation services each year, one near the close of the summer session, and one after the close of the first semester. About thirty-five new members are initiated each year at these services.

Six open meetings are held, at which topics of general mathematical interest are presented by guest speakers, faculty members, and students.

Topics presented by guest speakers were:

Mathematics in the secondary school, by Miss Jane Townsend, Principal of the Girard high school.

Vital Statistics, by D. E. Waggoner, Director of the Division of Vital Statistics for Kansas.

Topics presented by faculty and student members were:

Addition-subtraction logarithms, by Prof. F. C. German

The number 2 in mathematics, by James Hudson

Career fields for mathematicians, by Norval Phillips
The origin and meaning of mathematical terms, by Prof. J. A. G. Shirk
The scope of mathematics, by D. R. G. Smith.

A new conference room at the college provides adequate facilities for the public meetings to which all persons interested are cordially invited. Wives of veterans are finding these open meetings valuable in giving them a better understanding of the possible openings for their husbands after graduation.

Officers for 1947–48 are: President, James McCollam; Vice-President, James Hudson; Secretary, Betty Multhaup; Treasurer, Norval Phillips; Corresponding Secretary, Prof. J. A. G. Shirk; Sponsor, Dr. R. G. Smith.

Kappa Mu Epsilon, College of St. Francis

Meetings of the *Illinois Delta* Chapter of *Kappa Mu Epsilon* were held every month from October to May. At the various meetings the following papers were read:

An historical account of complex numbers, by Anne Whalen
Early forms of numerals and numerology, by Margaret Markiewicz
An historical account of fractions, by Sister M. Crescentia (visitor)
Introduction to the plane trigonometry of the Alamgest, by Marian Masters
Evolution of the symbolism in trigonometry, by Sister M. Claudia.

Other talks on book reports and recreational mathematics were given by members.

The officers for the year 1947–48 are: President, Elonore O'Connor; Vice-president, Betty Lanoue; Secretary, Jane Rourke; Treasurer, Mary Jean Lafond; Corresponding Secretary, Sister Rita Clare Flynn; Faculty Sponsor, Sister M. Claudia Zeller.

Pi Mu Epsilon, Michigan State College

The *Michigan Alpha* Chapter of *Pi Mu Epsilon* held eleven meetings in 1946–47, including Spring and Fall term picnics and a banquet in the Winter term. At three meetings a total of thirty one new members were initiated. Each initiate gave a brief talk on some mathematical riddle or fallacy or had to present a mathematical method for capturing lions. At most of the regular meetings two twenty minute talks were given by student members, after which refreshments were served. The following papers were presented:

The arbelos, by Eleanor Bessonen
The trimetric ruler, by Eunice Anderson
A problem in coin weighing, by Robert Jackson
Pathological functions, by Melvin LaVerne
Pathological curves, by Carolyn Trimm
A set of eight numbers, by Matha Hawley
Minimal tangents, by Nancy Pringle
A method for rapid calculation of Pi, by Mary Fuss
The five-color problem, by Phyllis Rowe.

Sixty members attended the annual banquet, and heard an address by Professor V. G. Grove on *A high school teacher of the nineteenth century*, in which he spoke of the life and work of the German mathematician, Karl Weierstrass.

The officers for 1946–47 were: President, Barbara Houston; Vice-President, Melvin LaVerne; Secretary-treasurer, Charles Costa; Membership Committee, Carolyn Trimm, Rosalie Manz, Phyllis Rowe; Faculty Sponsor, Prof. E. A. Nordhaus.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

NINTH CHRISTMAS CONFERENCE OF THE N.C.T.M.

The National Council of Teachers of Mathematics will hold its Ninth Christmas Conference at Ohio State University, Columbus, Ohio, on Wednesday and Thursday, December 29 and 30, 1948. Six sectional meetings will be held devoted to problems relating to the teaching of high school and junior college mathematics and the preparation and in-service training of teachers.

At the Thursday morning session, opportunities for discussion of important questions in mathematics teaching will be offered by attending one of seventeen discussion groups and clinics under the leadership of individual classroom teachers. Topics to be considered include questions involving the contents of courses in algebra and general mathematics, instructional and learning aids, guidance, tests, teaching of statistics in the high school and junior college, coordination of high school and college mathematics programs, and applications of mathematics in business and industry. The latest films and filmstrips will be shown at various periods during the two day conference.

Headquarters for the meeting will be in Baker Hall. Reservations for rooms in Baker Hall should be made by writing directly to Mr. Oscar Schaaf, Room 120 Arps Hall, Ohio State University, Columbus, Ohio not later than December 15, 1948.

A copy of the program may be obtained by writing to the National Council of Teachers of Mathematics, 212 Lunt Building, Northwestern University, Evanston, Illinois.

MECHANICS COLLOQUIUM AT ILLINOIS INSTITUTE OF TECHNOLOGY

The Mechanics Colloquium will consist of eight monthly lectures which will be held at Illinois Institute of Technology during the academic year 1948–1949.

Persons who desire an announcement of each session are asked to notify Mr. R. L. Janes, Illinois Institute of Technology, Technology Center, Chicago 16, Illinois.

PRELIMINARY ACTUARIAL EXAMINATIONS

The Actuarial Society of America and the American Institute of Actuaries offered prize awards to the nine undergraduates ranking highest on the combined score on Part 1 and Part 2 of the 1948 Preliminary Actuarial Examinations. The first prize of $200 was awarded to E. H. Larson, Massachusetts Institute of Technology. Additional prizes of $100 were awarded to each of the following: J. E. Brownlee, Haverford College; W. L. Farmer, University of Alabama; J. P. Fennell, Princeton University; B. F. Green, Jr., Yale University; Solomon Leader, Rutgers University; F. A. E. Pirani, University of Western Ontario; R. J. Semple, University of Toronto; C. A. Yardley, Dartmouth College.

The two actuarial organizations have authorized a similar set of nine prize awards for the 1949 Examinations on Part 2 which is a general mathematics examination covering algebra, trigonometry, coordinate geometry, differential and integral calculus. The closing date for applications is March 15, 1949.

Detailed information concerning the Examinations can be obtained from either of the following organizations: American Institute of Actuaries, 135 South La Salle Street, Chicago 3, Illinois; The Actuarial Society of America, 393 Seventh Avenue, New York 1, New York.

PERSONAL ITEMS

Professor E. F. Beckenbach of the University of California at Los Angeles represented the Association at the inauguration of President F. D. Fogg, Jr. of University of Southern California on June 11, 1948.

Professor Ruth M. Peters of St. Lawrence University was appointed a representative of the Association at the inauguration of President J. H. Davis of Clarkson College of Technology on October 8, 1948.

Dr. Miriam C. Ayer of Wellesley College has received an A.A.U.W. fellowship for study during 1948–49. She will return to Wellesley College in September, 1949 with the title of Assistant Professor.

Professor L. E. J. Brouwer of the University of Amsterdam has been elected to membership in the Royal Society of London.

Professor Daniel Buchanan was awarded an honorary degree of Doctor of Science by the University of British Columbia on the occasion of his retirement from the position of Dean of the Faculty of Arts and Science and Head of the Department of Mathematics.

Emeritus Professor Lennie P. Copeland of Wellesley College received an honorary degree of Doctor of Science from the University of Maine in June, 1948.

Professor Marston Morse of the Institute for Advanced Study has been elected a member of the Academia delle Scienze dell'Istituto di Bologna. Professor Morse has received recently an honorary degree of Doctor of Science from Kenyon College.

Professor Oswald Veblen of the Institute for Advanced Study has been elected to membership in the Accademia Nazionale dei Lincei.

Professor Hermann Weyl of the Institute for Advanced Study has been elected to honorary membership in the Calcutta Mathematical Society.

Albion College makes the following announcements: Professor H. D. Larsen has been appointed Head of the Department of Mathematics; Mr. Ralph Powers has been appointed to an instructorship.

At Stanford University a new Department of Statistics will be instituted this fall. Assistant Professor A. H. Bowker will serve as acting head of the department. Dr. M. A. Girshick, now research statistician for the government research project RAND at the Douglas Aircraft Company, has been appointed Professor of Statistics.

The University of Washington announces the following: Associate Professor C. M. Cramlet has been promoted to a professorship; Assistant Professor R. A. Beaumont has been promoted to an associate professorship; Professor T. G. Room of the University of Sydney has been appointed Visiting Professor for the Autumn Quarter, 1948; Dr. Edwin Hewitt of the University of Chicago has been appointed to an assistant professorship; Professor Nien-Yee Tang of Chias-Tung University has been appointed Lecturer; Dr. R. W. Ball of the University of Illinois, Mr. D. B. Dekker of the University of California have been appointed to instructorships.

Wellesley College announces the following appointments for 1948–49: Miss Zung-Nyi Loh as lecturer; Dr. Ilse Novak as instructor.

Professor C. B. Allendoerfer, who is on sabbatical leave from Haverford College for the academic year 1948–49, is at the Institute for Advanced Study.

Mr. J. W. Beach of Iowa State College of Agriculture and Mechanic Arts has been appointed to an associate professorship at the New Mexico School of Mines.

Mr. R. L. Beinert of Cornell University has been appointed to an instructorship at Hobart College.

Raj Chandra Bose has resigned as head of the graduate Department of Statistics of the University of Calcutta and has been appointed Professor of Mathematical Statistics at the University of North Carolina beginning in the winter of 1949.

Professor J. W. Bradshaw of the University of Michigan has retired with the title of Professor Emeritus.

Mr. G. P. Burns, who has been employed as a research physicist at the Naval Research Laboratory in Washington, has been appointed Assistant Professor of Physics at the Mary Washington College of the University of Virginia.

Associate Professor C. L. Buxton of Clarkson College of Technology has been made President of Paul Smith's College of Arts and Sciences.

Lieutenant P. F. Byrd, United States Army Air Corps, has been appointed to a professorship at Fisk University.

Assistant Professor R. L. Calvert of Utah State Agricultural College has been promoted to an associate professorship. He is on leave of absence for the year 1948–1949 and is resident at the University of Illinois.

Miss Louise H. Chin of the University of California has been appointed to an assistant professorship at the University of Arizona.

Dr. F. E. Clark has received an appointment as instructor at Tulane University.

Mr. Edward Craig has been appointed to an instructorship at Union College.

Professor J. C. Currie of Louisiana State University has accepted an appointment as associate professor at Alabama Polytechnic Institute.

Miss Alice C. Dean of Rice Institute has retired.

Assistant Professor Roy Dubisch of Triple Cities College, Syracuse University, has been appointed to an associate professorship at Fresno State College.

Mr. C. W. Dunnett has been appointed to an instructorship at the New York State Maritime Academy in Fort Schuyler.

Dr. W. F. Eberlein, who has been at the Institute for Advanced Study, has been appointed Assistant Professor of Mathematics at the University of Wisconsin.

Professor Beno Eckmann of the University of Lausanne has been appointed to a professorship at the Swiss Federal School of Technology in Zurich.

Miss Frances E. Falvey, formerly of Hollins College, has been appointed Dean of Women and Assistant Professor of Mathematics at James Millikin University.

Mr. Walter Fleming, lecturer at the University of Manitoba, has been appointed to an assistant professorship at Fort Hays Kansas State College.

Mr. M. K. Fort, Jr. of the University of Virginia has accepted an instructorship at the University of Illinois.

Mr. G. E. Gourrich has accepted a position as electrical engineer with the National Bureau of Standards.

Miss Helen Hand has been appointed to an instructorship at D'Youville College.

Dr. Frank Harary of the University of California has received an appointment as instructor at the University of Michigan.

Professor D. R. Hartree, Cambridge University has been granted a leave of absence to serve as acting director of the Institute for Numerical Analysis recently established at the University of California at Los Angeles as one of the operating units of the Applied Mathematics Laboratories, National Bureau of Standards.

Mr. T. R. Humphreys of Bergen Junior College has been appointed to an assistant professorship at New Jersey State Teachers College.

Professor W. R. Hutcherson, head of the Department of Mathematics and Astronomy of Berea College, has accepted a position as professor of mathematics at Northwestern State College, Natchitoches, Louisiana.

Mr. S. J. Jasper of the University of Kentucky has been appointed to an assistant professorship at Kent State University.

Assistant Professor J. R. F. Kent of the University of British Columbia has been appointed Associate Professor in Charge of Mathematics at Triple Cities College of Syracuse University.

Mr. Joseph Levitt of the University of Illinois is now Instructor in Physics at Pratt Institute.

Professor M. S. MacPhail of Acadia University, Canada, has been appointed to an associate professorship at Carleton College.

Professor A. W. McGaughey, Westminster College, has been appointed Associate Professor of Mathematics at Bradley University.

Associate Professor Josephine M. Mitchell of Oklahoma Agricultural and Mechanical College has received an appointment as assistant professor at the University of Illinois.

Assistant Professor Isaak Opatowski of the University of Michigan has accepted an appointment as research associate in mathematical biology at the University of Chicago.

Dean J. R. Overman of Bowling Green State University has retired from his position as Dean of the College of Liberal Arts but will continue as Professor of Mathematics.

Mr. I. B. Perrott of the College of Technology in Leicester has been appointed Lecturer at the University of Leeds.

Mr. O. L. Phillips, Louisiana State University, has been appointed Acting Head of the Department of Mathematics of Mississippi Southern College.

Mr. Peter Radkowski has accepted an instructorship at Washington and Jefferson College.

Professor R. F. Rinehart of Case Institute of Technology has been appointed Director of Planning Division, Research and Development Branch, Washington, D. C.

Mr. J. H. Ryser, University of Wisconsin, is now at the Institute for Acvanced Study.

Mr. N. C. Scholomiti of De Paul University has been appointed to an instructorship at the University of Illinois, Navy Pier, Chicago.

Dr. J. P. Sholz, who has been employed as mathematician by the Bartol Research Foundation of Franklin Institute, has been named Instructor in Mathematics and German at Wilson College.

Professor Emeritus E. R. Sleight of Albion College has been appointed Visiting Lecturer in Mathematics at the University of Richmond.

Dean H. L. Slobin of the University of New Hampshire has retired.

Associate Professor C. G. Stipe of the Michigan College of Mining and Technology has been promoted to a professorship.

Dr. R. E. Street of the University of New Mexico has been appointed Acting Professor of Aeronautical Engineering at the University of Washington.

Dean Elijah Swift of the College of Arts and Sciences, University of Vermont, has retired.

Professor Mary E. Wells, Vassar College, has retired.

Dr. A. L. Whiteman of the Office of Chief of Naval Operations, Washington, D. C. has been appointed to an assistant professorship at the University of Southern California.

Mr. R. W. Young, Lehigh University, has accepted an instructorship at the University of Florida.

Associate Professor J. W. T. Youngs of Indiana University has been promoted to a professorship.

Professor W. B. Campbell of Philadelphia Textile Institute died on August 12, 1948.

Reverend Paul Muehlmann of West Baden College died on August 28, 1948. He had been a member of the Association for twenty-five years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTIETH SUMMER MEETING OF THE ASSOCIATION

The thirtieth summer meeting of the Mathematical Association of America was held at the University of Wisconsin, Madison, Wisconsin, on Monday and Tuesday, September 6–7, 1948, in conjunction with the summer meeting and colloquium of the American Mathematical Society, and meetings of the Institute of Mathematical Statistics, the Econometric Society, and Section A of the American Association for the Advancement of Science. About seven hundred and seventeen persons were in attendance at the meetings, including the following three hundred and twenty members of the Association:

C. R. ADAMS, Brown University
V. W. ADKISSON, University of Arkansas
R. P. AGNEW, Cornell University
A. A. ALBERT, University of Chicago
FLORENCE E. ALLEN, University of Wisconsin
C. B. ALLENDOERFER, Institute for Advanced Study
K. J. ARNOLD, University of Wisconsin
W. L. AYRES, Purdue University
H. M. BACON, Stanford University
R. H. BARDELL, University of Wisconsin
J. J. BARRON, Marshall College
R. C. F. BARTELS, University of Michigan
WALTER BARTKY, University of Chicago
H. G. H. BARTRAM, University of Colorado
M. A. BASOCO, University of Nebraska
LEON BATTIG, University of Wisconsin in Milwaukee
R. A. BAUMGARTNER, Senior High School, Freeport, Illinois
HELEN P. BEARD, Newcomb College
H. M. BEATTY, Ohio State University

E. F. BECKENBACH, University of California at Los Angeles
E. G. BEGLE, Yale University
J. H. BELL, Michigan State College
A. A. BENNETT, Brown University
DOROTHY L. BERNSTEIN, University of Rochester
LIPMAN BERS, Syracuse University
H. R. BEVERIDGE, Monmouth College
T. A. BICKERSTAFF, University of Mississippi
F. C. BIESELE, University of Utah
R. H. BING, University of Wisconsin
LOLLIE BELLE BIENVENU, Louisiana State University
MAY C. BLACKSTOCK, University of Tennessee
J. W. BRADSHAW, University of Michigan
RICHARD BRAUER, University of Toronto
R. W. BRINK, University of Minnesota
FOSTER BROOKS, Kent State University
M. C. BROWN, University of Kentucky
R. H. BRUCK, University of Wisconsin
H. E. BUCHANAN, Tulane University
R. C. BUCK, Brown University
P. B. BURCHAM, University of Missouri
F. J. H. BURKETT, Union College
L. P. BURTON, Waysata, Minneota
L. E. BUSH, College of St. Thomas
J. H. BUSHEY, Hunter College
JEWELL HUGHES BUSHEY, Hunter College
W. H. BUSSEY, University of Minnesota
NIKOLINE A. BYE, Central Michigan College of Education
S. S. CAIRNS, University of Illinois
W. D. CAIRNS, Oberlin College
R. H. CAMERON, University of Minnesota
C. C. CAMP, University of Nebraska
R. E. CARR, Michigan State College
W. B. CARVER, Cornell University
J. O. CHELLEVOLD, Lehigh University
R. V. CHURCHILL, University of Michigan
F. MARION CLARKE, University of Nebraska
HELEN E. CLARKSON, Creighton University
NATHANIEL COBURN, University of Michigan
C. J. COE, University of Michigan
H. J. COHEN, University of Wisconsin
B. H. COLVIN, University of Wisconsin
E. G. H. COMFORT, Illinois Institute of Technology
H. H. CONWELL, Beloit College
V. F. COWLING, Lehigh University
C. C. CRAIG, University of Michigan
A. B. CUNNINGHAM, West Virginia University
J. H. CURTISS, National Bureau of Standards
E. H. CUTLER, Lehigh University
J. A. DAUM, Texas A. & M. College
P. H. DAUS, University of California at Los Angeles
ROBERT DAVIES, University of Wisconsin
D. R. DAVIS, State Teachers College, Montclair, New Jersey
JAMES ELMER DAVIS, Drexel Institute of Technology
REV. L. A. V. DECLEENE, Catholic University
A. H. DIAMOND, Oklahoma A. & M. College
L. L. DINES, Northwestern University
H. L. DORWART, Washington and Jefferson College
D. M. DRIBIN, Army Security Agency
NELSON DUNFORD, Yale University
W. L. DUREN, JR., Tulane University
W. H. DURFEE, Dartmouth College
P. S. DWYER, University of Michigan
W. F. EBERLEIN, University of Wisconsin
P. D. EDWARDS, Ball State Teachers College
MARGARET C. EIDE, State Teachers College, River Falls, Wisconsin
SAMUEL EILENBERG, Columbia University
W. S. ERICKSEN, Forest Products Laboratory
R. L. ERICKSON, Purdue University
H. P. EVANS, University of Wisconsin
H. S. EVERETT, University of Chicago
G. M. EWING, University of Missouri
A. B. FARNELL, Princeton University
J. V. FINCH, Chicago, Illinois
C. H. FISCHER, University of Michigan
L. R. FORD, Illinois Institute of Technology
J. S. FRAME, Michigan State College
EVELYN FRANK, University of Illinois
R. E. GASKELL, Iowa State College
C. B. GASS, Wesleyan University
H. M. GEHMAN, University of Buffalo
B. E. GILLAM, Drake University
J. W. GIVENS, University of Tennessee
A. M. GLEASON, Harvard University
R. A. GOOD, University of Maryland
RUTH E. GOODMAN, Duquesne University
CORNELIUS GOUWENS, Iowa State College
A. A. GRAU, Grand Rapids, Michigan
L. M. GRAVES, University of Chicago
W. L. GRAVES, Kansas State College
EDISON GREER, Kansas State College
V. G. GROVE, Michigan State College
S. G. HACKER, State College of Washington
N. A. HALL, University of Minnesota

P. R. HALMOS, University of Chicago
E. S. HAMMOND, Bowdoin College
A. G. HANSEN, Cleveland, Ohio
E. G. HARRELL, State Teachers College, Platteville, Wisconsin
W. L. HART, University of Minnesota
G. E. HAY, University of Michigan
M. B. HASLAM, University of Buffalo
NOLA L. HAYNES, University of Missouri
C. T. HAZARD, Purdue University
ANNA S. HENRIQUES, University of Utah
FRITZ HERZOG, Michigan State College
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
J. J. L. HINRICHSEN, Iowa State College
D. L. HOLL, Iowa State College
D. G. HORVITZ, Iowa State College
A. S. HOUSEHOLDER, Clinton Laboratories
R. C. HUFFER, Beloit College
RALPH HULL, Purdue University
P. F. HULTQUIST, University of Wisconsin
M. GWENETH HUMPHREYS, Newcomb College
J. W. HURST, Montana State College
M. A. HYMAN, Naval Ordnance Laboratory
M. H. INGRAHAM, University of Wisconsin
W. C. JANES, Kansas State College
H. F. S. JONAH, Purdue University
G. K. KALISCH, University of Minnesota
WILFRED KAPLAN, University of Michigan
IRVING KAPLANSKY, Institute for Advanced Study
SAMUEL KARLIN, California Institute of Technology
M. W. KELLER, Purdue University
CLARIBEL KENDALL, University of Colorado
K. F. KENNEY, University of Wisconsin at Milwaukee
E. C. KIEFER, James Milliken University
J. R. KLINE, University of Pennsylvania
JOHN KRONSBEIN, Hoosier-Cardinal Corporation
R. E. LANGER, University of Wisconsin
LEO LAPIDUS, Michigan State College
E. H. LARQUIER, Spring Hill College
H. D. LARSEN, Albion College
C. G. LATIMER, Emory University
W. G. LEAVITT, University of Nebraska
H. L. LEE, University of Tennessee
SOLOMON LEFSCHETZ, Princeton University
WALTER LEIGHTON, Washington University
W. J. LEVEQUE, University of Colorado
HARRY LEVY, University of Illinois
F. A. LEWIS, University of Alabama
CHARLES LOEWNER, Syracuse University
W. S. LOUD, University of Minnesota
C. C. MACDUFFEE, University of Wisconsin
SAUNDERS MACLANE, University of Chicago
H. W. MARCH, University of Wisconsin
MORRIS MARDEN, University of Wisconsin at Milwaukee
A. M. MARK, University of Wisconsin
ANNA MARM, Bethany College
M. H. MARTIN, University of Maryland
W. T. MARTIN, Massachusetts Institute of Technology
C. W. MATHEWS, Washington University
KENNETH MAY, Carleton College
J. R. MAYOR, University of Wisconsin
R. B. MCCLENON, Grinnell College
DOROTHY MCCOY, Belhaven College
EDITH A. MCDOUGLE, University of Delaware
W. H. MCEWEN, University of Manitoba
W. R. MCEWEN, University of Minnesota
L. H. MCFARLAN, University of Washington
J. V. MCKELVEY, Iowa State College
MARTHA M. MCKELVEY, Ames, Iowa
A. E. MEDER, JR., Rutgers University
A. S. MERRILL, Montana State University
B. E. MESERVE, University of Illinois
H. J. MISER, Williams College
B. E. MITCHELL, Millsaps College
T. W. MOORE, U. S. Naval Academy
MAX MORRIS, Case Institute of Technology
DOROTHY J. MORROW, Civil Aeronautics Administration
D. S. MORSE, Union College
LEO MOSER, University of Manitoba
E. J. MOULTON, Northwestern University
A. L. NELSON, Wayne University
W. K. NELSON, University of Colorado
W. J. NEMEREVER, University of Michigan
C. V. NEWSOM, University of the State of New York
GEORGE OGAWA, University of Chicago
RUFUS OLDENBURGER, DePaul University
EMMA J. OLSON, Kent State University
F. W. OWENS, Pennsylvania State College
HELEN B. OWENS, Pennsylvania State College
J. C. OXTOBY, Bryn Mawr College
GORDON PALL, Illinois Institute of Technology
S. T. PARKER, Kansas State College
W. V. PARKER, University of Georgia

G. A. PARKINSON, University of Wisconsin at Milwaukee
P. M. PEPPER, University of Notre Dame
C. R. PERISHO, Nebraska Wesleyan University
H. P. PETTIT, Marquette University
C. R. PHELPS, Rutgers University
GEORGE PIRANIAN, University of Michigan
A. E. PITCHER, Lehigh University
GERTRUDE V. PRATT, Central Michigan College
G. B. PRICE, University of Kansas
A. L. PUTNAM, University of Chicago
E. D. RAINVILLE, University of Michigan
J. F. RANDOLPH, University of Rochester
RUTH B. RASMUSEN, Chicago City College
G. E. RAYNOR, Lehigh University
M. O. READE, University of Michigan
C. J. REES, University of Delaware
MINA S. REES, Office of Naval Research
P. V. REICHELDERFER, Ohio State University
IRMA MOSES REINER, University of Illinois
ERIC REISSNER, Massachusetts Institute of Technology
J. G. RENNO, JR., University of Wisconsin
C. N. REYNOLDS, West Virginia University
C. E. RHODES, Alfred University
P. R. RIDER, Washington University
R. F. RINEHART, Research & Development Board
FRED ROBERTSON, Iowa State College
H. A. ROBINSON, Agnes Scott College
G. F. ROSE, University of Wisconsin
ARTHUR ROSENTHAL, Purdue University
J. B. ROSSER, Cornell University
W. E. ROTH, Sampson College
E. H. ROTHE, University of Michigan
LULU L. RUNGE, University of Nebraska
ARTHUR SARD, Queens College
A. C. SCHAEFFER, Purdue University
ROBERT SCHATTEN, University of Kansas
HENRY SCHEFFÉ, Columbia University
E. D. SCHELL, U. S. Bureau of Labor Statistics
EDITH R. SCHNECKENBURGER, University of Buffalo
I. J. SCHOENBERG, University of Pennsylvania
K. C. SCHRAUT, University of Dayton
AUGUSTA SCHURRER, University of Wisconsin
NATHAN SCHWID, University of Wyoming
E. J. SCOTT, University of Illinois
W. T. SCOTT, Northwestern University
COL. W. E. SEWELL, Information and Education Division, U. S. Army
M. E. SHANKS, Purdue University
I. M. SHEFFER, Pennsylvania State College
L. W. SHERIDAN, College of St. Thomas
L. S. SHIVELEY, Ball State Teachers College
SISTER MARIOLA DOBBIN, Rosary College
SISTER MARY FELICE, Mount Mary College
SISTER MARY FERRER MCFARLAND, University of Notre Dame
M. F. SMILEY, Northwestern University
DOROTHY B. SMITH, University of Oklahoma
F. C. SMITH, College of St. Thomas
G. W. SMITH, University of Kansas
H. L. SMITH, Louisiana State University
D. W. SNADER, Galesburg University
ERNEST SNAPPER, University of Southern California
ANDREW SOBCZYK, Watson Laboratories
ELIZABETH S. SOKOLNIKOFF, University of Wisconsin
T. H. SOUTHARD, Wayne University
F. W. SPARKS, Texas Technological College
LT. W. K. SPEARS, Ft. Bliss, Texas
R. H. STARK, Northwestern University
H. E. STELSON, Michigan State College
C. F. STEPHENS, Morgan State College
R. P. STEPHENS, University of Georgia
B. M. STEWART, Michigan State College
M. H. STONE, University of Chicago
E. B. STOUFFER, University of Kansas
A. G. SWANSON, Gustavus Adolphus College
OTTO SZASZ, University of Cincinnati
A. H. TAUB, University of Washington
J. S. TAYLOR, University of Pittsburgh
H. P. THIELMAN, Iowa State College
J. M. THOMAS, Duke University
R. M. THRALL, University of Michigan
W. J. THRON, Washington University
JOHN TODD, King's College, London
LEONARD TORNHEIM, University of Michigan
J. I. TRACEY, Yale University
G. R. TROTT, University of Mississippi
P. L. TRUMP, University of Wisconsin
A. W. TUCKER, Princeton University
BRYANT TUCKERMAN, Cornell University
J. W. TUKEY, Princeton University
P. M. TULLIER, Loyola University
H. L. TURRITTIN, University of Minnesota
J. I. VASS, University of Wisconsin
JOHN VON NEUMANN, Institute for Advanced Study
R. W. WAGNER, Oberlin College
G. L. WALKER, Purdue University
R. J. WALKER, Cornell University

J. L. WALSH, Harvard University
J. A. WARD, University of Georgia
K. W. WEGNER, Carleton College
MARIE J. WEISS, Newcomb College
R. L. WESTHAFER, New Mexico College of A & M
A. E. WHITFORD, Alfred University
P. M. WHITMAN, Johns Hopkins University
L. R. WILCOX, Illinois Institute of Technology
R. L. WILDER, University of Michigan
J. E. WILKINS, JR., American Optical Company
S. S. WILKS, Princeton University
R. L. WILSON, University of Tennessee
R. M. WINGER, University of Washington
LOUISE A. WOLF, University of Wisconsin
F. H. YOUNG, Oregon State College
J. W. T. YOUNGS, Indiana University
J. L. ZEMMER, JR., University of Wisconsin
M. A. ZORN, Indiana University

Rooms for members of the mathematical organizations and their families were provided in Adams Hall and Tripp Hall, and meals were served in the cafeteria in Van Hise Hall. Tea was served by the ladies of the Department of Mathematics of the University of Wisconsin on Tuesday afternoon on the lawn beside Lake Mendota. That evening a concert of chamber music was presented in the Music Hall Auditorium by the *Pro Arte* quartet of the University of Wisconsin consisting of Rudolph Kolisch, first violin; Albert Rahier, second violin; Bernard Milofsky, viola; and Ernst Friedlander, cello.

An opportunity was given on Tuesday evening and again on Thursday evening to visit the Washburn Astronomical Observatory. A film on "Solar Explosions" was shown and visitors were permitted to inspect the telescope of the Observatory.

Wednesday afternoon was left open for recreation. After the group picture was taken in the Stock Pavilion, members could choose between a boat ride around Lake Mendota, a visit to the U. S. Forest Products Laboratory, or a visit to the Headquarters of the U. S. Armed Forces Institute. On Thursday evening, a picnic was held at Picnic Point, a scenic spot about a mile distant from the dormitories.

A dinner for members of the mathematical organizations was held on Wednesday evening in the Crystal Ball Room of the Hotel Loraine. Dean Walter Bartky was the toastmaster, and Dean M. H. Ingraham welcomed the guests on behalf of the University of Wisconsin. Short talks were given by Professor J. L. Walsh, Dr. C. V. Newsom, and Professor P. S. Dwyer on behalf of the Society, the Association, and the Institute. In conclusion, Dean Bartky expressed the thanks of the visitors for the excellent work of the local members of the Committee on Arrangements, especially Professor H. P. Evans, Chairman, and Professors B. H. Colvin and Elizabeth S. Sokolnikoff. A formal resolution to the same effect prepared by Professor G. B. Price was adopted at the meeting on Thursday afternoon.

The sessions of the American Mathematical Society began on Tuesday afternoon and continued through Friday morning. The colloquium lectures on "Representations of Groups and Rings" were given by Professor Richard Brauer of the University of Toronto. On Thursday afternoon, Professor J. W. T. Youngs of Indiana University delivered an address entitled "Topological meth-

ods in the theory of Lebesgue area." On Wednesday morning the Society held a joint session with Section A of the A.A.A.S. at which time Professor R. L. Moore of the University of Texas gave his retiring address as Vice-President of the A.A.A.S. and Chairman of Section A on the subject of "Spirals."

Sessions of the Institute of Mathematical Statistics and the Econometric Society were held from Tuesday morning through Friday morning. A joint session on stochastic processes was held on Wednesday and Thursday was devoted to a symposium on the theory of games. On Friday morning there was a joint session on sequential estimation.

The Mathematical Association held its sessions on Monday afternoon and Tuesday morning in Room 272 of Bascom Hall. President L. R. Ford presided at the first session and Vice-President C. B. Allendoerfer at the second session. The program was arranged by a committee consisting of W. L. Ayres, Chairman, R. H. Bing, and D. H. Lehmer.

FIRST SESSION OF THE ASSOCIATION

"Mathematical problems associated with the use of laminates in aircraft," by Professor H. W. March, University of Wisconsin and U. S. Forest Products Laboratory.

"Definition of angles in higher dimensional spaces," by Professor S. S. Cairns, University of Illinois.

"An experiment in teaching large classes in mathematics," by Professor H. F. S. Jonah, Purdue University.

SECOND SESSION OF THE ASSOCIATION

"Exterior ballistics of artillery rockets," by Professor J. B. Rosser, Cornell University.

"What is homotopy?" by Professor Samuel Eilenberg, Columbia University.

"Enrichment of the mathematical curriculum for juniors and seniors," by Professor M. H. Stone, University of Chicago.

MEETING OF THE BOARD OF GOVERNORS

The Board met at 7:30 p.m. on Monday in the Lounge of Slichter Hall. Of the thirty-five members of the Board, twenty-four were present. Following are some of the more important items of business transacted.

The Board approved the appointment of L. L. Dines in place of G. C. Evans as a member of the Nominating Committee for 1948.

It was voted to hold a meeting of the Association in June 1949 at Rensselaer Polytechnic Institute, Troy, New York, in conjunction with the annual meeting of the American Society for Engineering Education. The regular summer meeting of the Association will be held in Boulder, Colorado, in September 1949. The Board also voted to hold no meeting of the Association in the summer of

1950, since the International Congress of Mathematicians will be held at Harvard University at that time.

Caroline A. Lester and L. J. Green were elected Associate Editors of the MONTHLY and the resignation of R. F. Rinehart as an Associate Editor was accepted.

The Board voted as a general policy to pay an honorarium to authors of Slaught Memorial Papers at the rate of one dollar per printed page.

Effective December 1, 1948, the prices of all Carus Monographs are to be raised to $1.75 for members and to $3.00 for copies purchased through the Open Court Publishing Company.

An appropriation of $1000 was voted from the General Fund of the Association toward the expenses of the International Mathematical Congress to be held in 1950.

The Board voted that the Association accept the invitation to be officially represented on the Policy Committee for Mathematics by the appointment of three members. The president was authorized to appoint temporary representatives to serve until the next annual meeting of the Board.

It was agreed that the Association should resume the direction of the Putnam Prize Competition and that the President be authorized to appoint a Director of Putnam Competition for a term of five years.

The Board voted unanimously to recommend to the Association that W. B. Carver be elected an honorary life member. At a brief business meeting of the Association held on Tuesday morning, this recommendation was approved and Professor Carver became the eighth person to be honored by election to honorary life membership.

MEETING OF SECTION SECRETARIES

A meeting of secretaries of the Sections of the Association was held at 7:30 p.m. on Tuesday in the Lounge of Slichter Hall. Of the twenty-five Sections, twenty-one were represented by their secretaries or other officers.

Associate Secretary Edith R. Schneckenburger sketched the history and activities of the sections from the founding of the Association. A statistical table giving data on membership and potential membership of each section was distributed.

Professor C. C. MacDuffee spoke on the work of the Committee for the Coordination of Studies in Mathematical Education, of which he is chairman.

A general discussion followed concerning the relations of the central office of the Association to the section secretaries. Matters of finance, procurement of speakers, and election of sectional governors were discussed for possible future action by the Board of Governors.

It is expected that similar meetings of section secretaries will be held at future meetings of the Association.

H. M. GEHMAN, *Secretary-Treasurer*

APRIL MEETING OF THE OHIO SECTION

The thirty-second annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Saturday, April 3, 1948. Professor H. S. Pollard, chairman of the Section, presided at the morning and afternoon sessions.

Ninety-two persons registered attendance, including the following seventy-two members of the Association: H. H. Alden, W. E. Anderson, Max Astrachan, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Theodore Bennett, W. D. Berg, J. B. Brandeberry, Foster Brooks, V. B. Caris, F. E. Carr, A. B. Carson, C. W. Cassel, B. B. Clark, Florentina M. Clinton, Wayne Dancer, R. C. Davis, R. H. Downing, B. B. Dressler, P. L. Evans, H. E. Fettis, B. E. Gatewood, Landis Gephart, B. C. Glover, L. J. Green, Ralph Hafner, Marshall Hall, R. G. Helsel, E. D. Jenkins, M. L. Johnson, Margaret E. Jones, K. D. Kelly, L. C. Knight, Jr., H. W. Kuhn, L. L. Lowenstein, R. H. Marquis, Margaret Mauch, S. W. McCuskey, H. E. Menke, Sister M. Mercedes, E. J. Mickle, L. H. Miller, C. C. Morris, Max Morris, Emma J. Olson, H. C. Parrish, C. G. Peckham, H. S. Pollard, Tibor Rado, S. E. Rasor, R. F. Rinehart, Louis Ross, S. A. Rowland, K. C. Schraut, C. E. Sealander, Samuel Selby, Ruth B. Smyth, W. H. Spragens, V. C. Stechschulte, C. W. Topp, E. P. Vance, R. H. Van Voorhis, R. W. Wagner, E. H. Wang, D. R. Whitney, F. B. Wiley, Alberta Wolfe, W. D. Wood, G. E. Woodson, Jr., C. H. Yeaton, Marie M. Yeaton.

The following officers were elected for the coming year: Chairman, R. H. Marquis, Ohio University; Secretary-Treasurer, Foster Brooks, Kent State University; Member of Executive Committee, L. H. Miller, Ohio State University; Member of Program Committee, I. A. Barnett, University of Cincinnati. The next regular meeting of the Section is scheduled to be held at the Ohio State University, Columbus, Ohio, on Saturday, April 2, 1949.

The following papers were presented:

1. *The mathematics major*, by Professor H. S. Pollard, Miami University.

The scientific developments which have taken place during and since the war have tended to emphasize the application of mathematics to engineering and science. There may be a tendency to regard the mathematics major as a potential engineer or scientist, and to confine his training to such topics as he will find of "practical" value in preparing him in the shortest time for further technical training.

The point of view of this paper is that the mathematics major is a composite of a number of primary interests and vocational aims, and that to stream-line courses for one particular group of students, particularly in a small institution where provision cannot be made for a number of different groups, is a fairly certain way of losing the interest of other students. Acceleration for accelerations's sake is not always desirable. The traditional subject matter and methods of mathematics still have value. In spite of the recent growth which mathematics has experienced in conjunction with the physical sciences, historically mathematics belongs to the liberal arts and is not a mere technical tool of the sciences. Mathematics is more than a collection of manipulative skills; it is a body of concepts and methods that constitute a way of thinking.

2. *A report on a recent questionnaire regarding mathematics in Ohio colleges*, by Professor Wayne Dancer, University of Toledo.

Most of the 35 colleges reporting encounter students with inadequate preparation in mathematics to the extent that they feel obliged to give courses in elementary and intermediate algebra, and in plane and solid geometry.

The most popular courses on the Junior-Senior level area, in order: Differential Equations, Theory of Equations, Advanced Calculus, Higher Algebra, and Mathematics of Statistics. Nearly half of the colleges give a four (semester) hour course in analytic geometry, including some solid analytics. The first course in calculus meets ordinarily 4 or 5 times a week. Twenty of the colleges teach differential and integral calculus alternately in a "unified" course. The author tabulated the topics which are most commonly covered in a course in Advanced Calculus.

It was found that most colleges are now giving special consideration to those non-technical students planning to take not more than a year of college mathematics. Very few colleges explicitly require mathematics for a bachelor's degree, but the majority do specify the study of mathematics or science. Half of the colleges reporting graduate fewer than six mathematics majors per year. The average number of hours required for a major is slightly in excess of 28, the minimum being 24, and the maximum 36.

3. *A coil winding problem*, by C. L. Emmerich, University of Cincinnati, introduced by Professor I. A. Barnett.

The problem is that of finding the manner of winding a coil having a fixed resistance R and maximum number of turns N. This may be formulated as an isoperimetric problem in the calculus of variations. It is found that the wire diameter is proportional to the square root of the distance from the coil axis.

4. *A solution of the problem of rapid scanning radar antenna*, by Professor R. F. Rinehart, Case Institute of Technology.

The problem of designing a method of focusing electromagnetic energy by electrical means is considered. The so-called rapid scanning problem in radar antenna is essentially the problem of designing a method for the efficient focusing of electromagnetic energy into a parallel beam by electrical means, and of providing for the rapid variation of the direction of this beam. The question reduces to the following problem in differential geometry: Determine a surface which contains a circle and a straight line segment (called the aperture) with respect to which the following two conditions are satisfied: (1) The circle is a geodesic circle relative to some point of the segment as center; (2) All geodesics joining *any* fixed point of the circle to all the points of the line segment cut the line segment at a fixed angle. Earlier efforts at solution of this problem were reviewed, and a solution in the form of a certain surface of revolution was described. This solution satisfactorily fulfills the requirements with the exception that the required line segment is a virtual, rather than an actual, aperture.

5. *Equations and loci in polar coördinates*, by Professor R. W. Wagner, Oberlin College.

This paper appeared in the June-July number of this MONTHLY, 1948.

6. *Matric diophantine equations*, by Professor I. A. Barnett, University of Cincinnati, and Professor C. W. Mendel, University of Illinois (introduced by Professor Barnett).

The authors consider equations of the form $X^k + Y^k = Z^k$, where the symbols X, Y, Z stand for

matrices whose elements may be zero, positive, or negative integers, and where multiplication is interpreted as matric multiplication. A general method of solution is given for the equation $X^2+Y^2=Z^2$ where the matrices are of order n, with Y and Z commutative, and X non-singular. For matrices of order 2, necessary and sufficient conditions for a solution are obtained when $k=2, 3, 4, 5$. These conditions are given in terms of the traces and determinants of the matrices involved. The equation $X^3+Y^3=I$ is fully discussed.

7. *Prospects in projective geometry*, by Professor Marshall Hall, The Ohio State University.

Professor Hall's paper was a one hour address by invitation of the Program Committee. The fundamental axioms of geometry are of three types: incidence, order, and metric axioms. Projective geometry is the study of the axioms of incidence. Ordinarily analytic geometry is thought of as giving coördinates to points in terms of distances to axes, and as such a part of metric geometry. Van Staudt first discovered in 1856 that a purely projective construction of coördinates could be made. In this construction the theorem of Desargues plays a central role. The theorem of Desargues is equivalent on the one hand to the existence of coördinates from a skew field, and on the other to the possibility of embedding a projective plane into a space of higher dimensions. The theorem of Pappus (or Pascal) implies the theorem of Desargues and also the commutative law of multiplication. Recent work has established more far-reaching relations between projective geometries and many branches of algebra, such as lattice theory, alternative rings, groups, and the theory of loops.

FOSTER BROOKS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

Thirty-first Summer Meeting, Boulder, Colorado, September, 1949.

The following is a list of the Sections of the Association with dates of future meetings insofar as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Bradley University, Peoria, May 13–14, 1949

INDIANA, University of Notre Dame, Spring, 1949

IOWA, Drake University, Des Moines, April 15–16, 1949

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, Spring, 1949

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA

MISSOURI

NEBRASKA, Lincoln, May, 1949

NORTHERN CALIFORNIA, San Francisco, January 29, 1949

OHIO, Ohio State University, Columbus, April 2, 1949

OKLAHOMA

PACIFIC NORTHWEST, Oregon State University, Corvallis, Spring, 1949

PHILADELPHIA, Philadelphia, November 27, 1948

ROCKY MOUNTAIN, Colorado School of Mines, Golden, April, 1949

SOUTHEASTERN, University of Alabama, University, March 18–19, 1949

SOUTHERN CALIFORNIA, John Muir College, Pasadena, March 12, 1949

SOUTHWESTERN

TEXAS, Denton, Spring, 1949

UPPER NEW YORK STATE, University of Buffalo, May, 1949

WISCONSIN, Lawrence College, Appleton, May 14, 1949

Books of Meaning

SCIENCE AT WAR

By J. G. Crowther and R. Whiddington. The first detailed account of science's contribution to the war effort, based on the official archives and documents assembled by the Scientific Advisory Committee to the British Cabinet. Because of the close and intimate collaboration between the American and British forces during World War II, this volume is of fundamental interest to the American scientist, as well as to the intelligent layman interested in science. $6.00

SCIENCE SINCE 1500

By H. T. Pledge, Librarian, Science Museum of London. "A thorough-going history of science. The author takes the subject and the reader in his stride on a journey which must be ranked as a very able performance indeed. The illustrations are excellent, the bibliography is unusually helpfully arranged, and the indices are full and satisfactory." —*M. F. Ashley Montagu, The New York Times.* $5.00

DICTIONARY OF SCIENCE AND TECHNOLOGY—*In 4 Languages*

By Maxim Newmark. "In the first 200 pages of this useful, practical dictionary are listed some 10,000 current English terms, each with its French, German and Spanish equivalents. The words listed are those most frequently used today in the physical sciences and mathematics. Finally, there are several pages of conversion tables and lists of technical abbreviations for each language. This dictionary should prove to be exceedingly useful. The library of every scientific and technical department should have a copy available."—*American Journal of Physics.* $6.00

PHYSICS OF THE 20TH CENTURY

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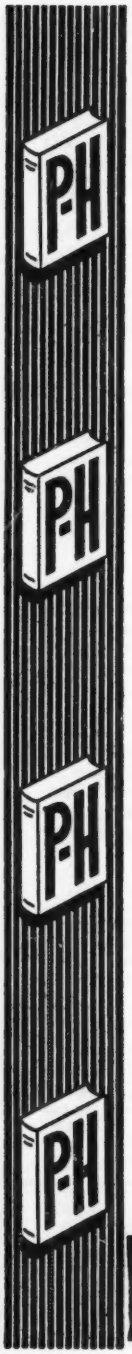

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